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Witten Laplacian Method for The Decay of Correlations

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Abstract The aim of this paper is to apply direct methods to the study of integrals that appear naturally in Statistical Mechanics and Euclidean Field Theory. We provide weighted estimates leading to the exponential decay of the two-point correlation functions for certain classical convex unbounded models. The methods involve the study of the solutions of the Witten Laplacian equations associated with the Hamiltonian of the system.

Keywords Witten Laplacians · Statistical mechanics correlations · Exponential decay

1 Introduction

In this paper, we study partial differential equation techniques for problems coming from equilibrium Statistical Mechanics and Euclidean Field theory. In the context of classical equilibrium Statistical Mechanics, one is interested in a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components.

We shall consider systems where each component is located at a site i of a crystal lattice $\Lambda \subset \mathbb{Z}^d$, and is described by a continuous real parameter $x_i \in \mathbb{R}$. A particular configuration of the total system will be characterized by an element $x = (x_i)_{i\in\Lambda}$ of the product space $\Omega = \mathbb{R}^\Lambda$. This set Ω is called the configuration space or phase space.

We shall denote by $\Phi = \Phi^\Lambda$ the Hamiltonian which assigns to each configuration $x \in \mathbb{R}^\Lambda$ a potential energy $\Phi(x)$. The probability measure that describes the equilibrium of the system is then given by the Gibbs measure

$$d\mu^\Lambda(x) = Z_\Lambda^{-1} e^{-\Phi(x)} dx.$$

$Z_\Lambda > 0$ is a normalization constant.

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For any finite domain Λ of $\mathbb{Z}^d$, we shall consider a Hamiltonian Φ of the phase space $\mathbb{R}^\Lambda$, satisfying conditions that will guarantee the solvability of the corresponding Witten Laplacian equations which we will introduce below. Namely:

1. $\lim_{|x|\to\infty} |\nabla\Phi(x)| = \infty$.
2. For some M, any $\partial^\alpha\Phi$ with $|\alpha| = M$ is bounded on $\mathbb{R}^\Lambda$.
3. For $|\alpha| \geq 1$, $|\partial^\alpha\Phi(x)| \leq C_\alpha(1 + |\nabla\Phi(x)|^2)^{1/2}$ for some $C_\alpha > 0$.
4. $\operatorname{Hess}\Phi \geq \delta$ for some $0 < \delta \leq 1$.

Here and in what follows, $\alpha = (\alpha_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ shall denote a multiindex. We set $|\alpha| = \sum_{i\in\Lambda}\alpha_i$, $\alpha! = \alpha_1!\cdots\alpha_n!$. If $\beta = (\beta_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ and $\beta_j \leq \alpha_j$ for all $j \in \Lambda$, then we write $\beta \leq \alpha$. For $\alpha,\ \beta \in \mathbb{Z}_+^{|\Lambda|}$ such that $\beta \leq \alpha$, we put $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. If $\alpha = (\alpha_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ and $x \in \mathbb{R}^\Lambda$ we write $x^\alpha = \prod_{i\in\Lambda} x^{\alpha_i}$, and $\partial^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1}\cdots\partial^{\alpha_m}/\partial x_m^{\alpha_m}$ where $m = |\Lambda|$. Finally, if i and j are two nearest neighbor sites in $\mathbb{Z}^d$ we write $i \sim j$.

Definition 1 The lattice support, S_g of a function g on $\mathbb{R}^\Lambda$ is defined here to be the smallest subset Γ of $\Lambda \subset \mathbb{Z}^d$ for which g can be written as function of x_l alone with $l \in \Gamma$. For instance, if $g = x_i$, $S_g = \{i\}$.

If g and h are suitable functions on $\mathbb{R}^\Lambda$ with lattice support $S_g, S_h \subset \Lambda$ respectively, we shall study the behavior of the covariance

$$\operatorname{cov}(g,h) = \langle (g - \langle g\rangle_\Lambda)(h - \langle h\rangle_\Lambda)\rangle_\Lambda$$

as $d(S_g, S_h) \to \infty$. Here $\langle\cdot\rangle_\Lambda$ is the ensemble average with respect to the Gibbs measure $Z_\Lambda^{-1}e^{-\Phi_\Lambda}dx$, and $d(\cdot,\cdot)$ is the usual distance in $\mathbb{Z}^d$.

In particular, if $g = x_i$ and $h = x_j$, we obtain the two-point correlation function

$$\operatorname{Cor}^\Lambda(i,j) = \langle (x_i - \langle x_i\rangle_\Lambda)(x_j - \langle x_j\rangle_\Lambda)\rangle_\Lambda.$$

This measures the correlation between local spin deviations occurring at the i-th and j-th sites.

The exponential decay of $\operatorname{Cor}^\Lambda(i,j)$ with respect to $d(i,j)$ was proved by Helffer and Sjöstrand [7] in the one dimensional case ($d = 1$) for models whose Hamiltonians are of the form

$$\Phi^{(m)}(x) = \frac{x^2}{2} + \Psi(x) \quad x \in \mathbb{R}^m,$$

with $\Phi^{(m)}$ satisfying hypotheses 1–4 above. They indeed proved the following theorem.

Theorem 1 (Helffer-Sjöstrand [7]) *Let*

$$\Phi = \Phi^{(m)}(x) = \frac{x^2}{2} + \Psi(x) \tag{1.1}$$

satisfy

$$|\partial^\alpha\nabla\Psi| \leq C_\alpha, \quad \forall\alpha \in \mathbb{Z}_+^m, \tag{1.2}$$

where $\alpha = (\alpha_1,\ldots,\alpha_m)$ *is a multiindex, and*

$$\operatorname{Hess}\Phi(x) \geq \delta, \quad \textit{for some } 0 < \delta < 1. \tag{1.3}$$

If in addition

$$\|\mathrm{Hess}\,\Phi(x)\|_{\mathcal{L}(\ell^\infty_\rho)} \le C, \tag{1.4}$$

(*where all the constants are independent of the size of the lattice ring if nothing else is indicated*) *for all* ρ *on* $\mathbb{Z}/m\mathbb{Z}$ *satisfying*

$$e^{-\kappa} \le \frac{\rho(i+1)}{\rho(i)} \le e^{\kappa}, \quad \textit{for some } \kappa > 0,$$

and

$$\|\mathrm{Hess}\,\Phi(x) - \mathbf{I}\|_{\mathcal{L}(\ell^\infty_\rho)} \le \delta < 1 \tag{1.5}$$

where ℓ^∞_ρ *is the weighted* ℓ^∞*-space defined by the norm*

$$|x|_{\ell^\infty_\rho} = \max_i |\rho(i) x_i|, \tag{1.6}$$

then

$$\left|\mathrm{Cor}^{(m)}(i,j)\right| \le C_\varepsilon e^{-(\kappa-\varepsilon) dist_{\mathbb{Z}/m\mathbb{Z}}(i,j)} \tag{1.7}$$

for all $m, \varepsilon > 0$ *and all pairs* (i, j).

The proof of this theorem strongly relies on the one dimensional situation. The abstract result in [7] was then illustrated by a mean field model introduced by Kac [9] whose Hamiltonian is given by

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{i=1}^{m} \ln\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_{i+1})\right] \tag{1.8}$$

with the convention $x_{m+1} = x_1$.

In this paper, we propose to extend the result of Theorem 1 to higher dimensions ($d > 1$) without introducing the technical assumptions

$$\|\mathrm{Hess}\,\Phi(x)\|_{\mathcal{L}(l^\infty_\rho)} \le C$$

and

$$\|\mathrm{Hess}\,\Phi(x) - \mathbf{I}\|_{\mathcal{L}(l^\infty_\rho)} \le \delta < 1.$$

We shall prove the following proposition.

Proposition 1 (The main result) *Let* Λ *be a subset of* $\mathbb{Z}^d$ $(d \ge 1)$. *If*

$$\Phi = \Phi^{(\Lambda)}(x) = \frac{x^2}{2} + \Psi(x) \quad x = (x_i)_{i\in\Lambda} \in \mathbb{R}^\Lambda$$

satisfies

$$\mathrm{Hess}\,\Phi(x) \ge \delta \quad \textit{for some } 0 < \delta < 1, \tag{1.9}$$

$$|\partial^\alpha \nabla\Psi| \le C_\alpha, \quad \forall \alpha \in \mathbb{Z}_+^{|\Lambda|} \tag{1.10}$$

for some $C_\alpha > 0$ *where* $\alpha = (\alpha_1, \dots, \alpha_{|\Lambda|})$, *and if there exists* $\delta_0 \in (0, 1)$ *such that*

$$\left\langle a, M^{-1} \operatorname{Hess} \Phi(x) M a \right\rangle \geq \delta_0 a^2 \quad \forall x \in \mathbb{R}^\Lambda, \forall a \in \mathbb{R}^\Lambda \tag{1.11}$$

where M *is the diagonal matrix*

$$M = \left(\delta_{ij} e^{-\kappa d(i, S_g)}\right)_{i,j \in \Lambda}$$

for some $\kappa > 0$, *then for any smooth functions* g *and* h *satisfying* (1.10) *on* $\mathbb{R}^{S_g}$, *and* $\mathbb{R}^{S_h}$ *with* $S_g \cap S_h = \varnothing$ (S_g *and* S_h *denote respectively the support of* g *and* h *as defined above*), *we have*

$$|\operatorname{cov}(g, h)| \leq C e^{-\kappa d(S_g, S_h)} \tag{1.12}$$

where C *and* κ *are positive constants that do not depend on* Λ, *but possibly dependent on the size of the supports of* g *and* h.

Remark 1 The proof of this proposition is based on a weighted estimate of the solution of the elliptic system

$$\begin{cases} -\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle_{L^2(\mu^\Lambda)}, \\ \langle f \rangle_{L^2(\mu^\Lambda)} = 0. \end{cases}$$

We shall use the fact that the solution f satisfies $\partial^\alpha \nabla f(x) \to 0$ as $|x| \to \infty$ $\forall \alpha \in \mathbb{Z}_+^{|\Lambda|}$. This argument on the asymptotic behavior of f will be proved only in the case where $\Phi(x) = \frac{x^2}{2} + \Psi(x)$ together with the assumption that both Ψ and g are compactly supported. The estimate will then be obtained in the general case by means of a family of cut-off functions.

The result of this proposition will be illustrated by the d-dimensional nearest neighbor Kac model, where the potential is given by

$$\Phi(x) = \frac{x^2}{2} - 2 \sum_{i,j \in \Lambda, i \sim j} \ln \cosh \left[\sqrt{\frac{\beta}{2}} \left(x_i + x_j\right) \right], \quad x = (x_i)_{i \in \Lambda} \in \mathbb{R}^\Lambda,$$

for $\beta > 0$ smaller than some value β_0 to be determined.

This result may also be viewed as an extension of some previous work of Sjöstrand-Bach-Jecko [3], Bach-Moller [2] and Antoniouk-Antoniouk [1]. On the exponential decay of the two-point correlations functions to a larger class of unbounded convex models whose Hamiltonian are of the form

$$\Phi(x) = \frac{x^2}{2} + \Psi(x).$$

As mentioned in [1], the study of the exponential decay of correlations is rather complicated in the cases of unbounded models because of the difficulties one might have to apply Dobrushin uniqueness technique. Many results in this direction were obtained in the case that the interaction potentials are quadratic. Antoniouk and Antoniouk [1] treated the problem of the exponential decay of correlations in the case of nonquadratic polynomial interactions with Hamiltonians of the type

$$H(x) = \sum_{i \in \mathbb{Z}^d} (1 + x_i^2)^{2n+1} + \lambda \sum_{i,j \in \mathbb{Z}^d} b_{i-j} (x_i - x_j)^{2n+2}.$$

Their methods are mainly based on Brascamp-Lieb inequality [5].

New methods have been recently developed for the study of the decay of correlations through the analysis of Witten Laplacians on one forms [2, 3] and [8]. In [3], the authors studied the exponential decay of correlations for models of the form

$$H_\Lambda(x) = \sum_{i\in\Lambda} f(x_i) + \lambda \sum_{i,j\in\Lambda} e^{-\upsilon d(i-j)} w_{ij}(x_i, x_j)$$

where $d(\cdot)$ is a distance function and f and w_{ij} are C^2 functions. (See [3] for more details.) In [2], Bach and Moller improved the results in [3] by studying models of the type

$$H_\Lambda(x) = \sum_{i\in\Lambda} f(x_i) + \lambda \sum_{i,j\in\Lambda} w_{ij}(x_i, x_j).$$

These authors treated the problem with several technical assumptions on Hamiltonians that are restricted only to two body interactions. In this paper, we consider a more restrictive one particle phase, but a more general many body potential. However, we believe that our result may be generalized to a wider class of Hamiltonians that are not necessarily of Kac type.

As in [3, 7] and [2], our method is generally based on the analysis of suitable differential operators

$$\mathbf{W}_\Phi^{(0)} = \left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}\right)$$

and

$$\mathbf{W}_\Phi^{(1)} = -\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} + \operatorname{Hess}\Phi.$$

These are in some sense deformations of the standard Laplace Beltrami operator. These operators, commonly called Witten Laplacians, were first introduced by Edward Witten [14] in 1982 in the context of Morse theory for the study of topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Johannes Sjöstrand [7] introduced two elliptic differential operators.

$$A_\Phi^{(0)} := -\Delta + \nabla\Phi\cdot\nabla$$

and

$$A_\Phi^{(1)} := -\Delta + \nabla\Phi\cdot\nabla + \operatorname{Hess}\Phi.$$

These operators provide direct methods for the study of integrals in high dimensions of the type that appear in Statistical Mechanics and Euclidean Field Theory. In 1996, J. Sjöstrand [12] observed that these so called Helffer-Sjöstrand operators are in fact equivalent to Witten's Laplacians. Indeed,

$$W_\Phi^{(.)} = e^{-\Phi/2} \circ A_\Phi^{(.)} \circ e^{\Phi/2} \tag{1.13}$$

and the map

$$U_\Phi : L^2(\mathbb{R}^\Lambda) \to L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx),$$
$$u \longmapsto e^{\frac{\Phi}{2}} u$$

is unitary.

There have been significant advances in the use of these Laplacians to study the thermodynamic behavior of quantities related to the Gibbs measure $d\mu^\Lambda = Z_\Lambda^{-1} e^{-\Phi} dx$. As a simple illustration, if one is interested in the study of the mean value $\langle g\rangle_\Lambda$, where

$$\langle g\rangle_\Lambda = \int g d\mu^\Lambda$$

and

$$d\mu^\Lambda = Z_\Lambda^{-1} e^{-\Phi_\Lambda} dx$$

for a suitable smooth function g with $g(0) = 0$, one can first solve the equation

$$\nabla g = (-\Delta + \nabla\Phi \cdot \nabla)\mathbf{v} + \text{Hess}\,\Phi\mathbf{v},$$

for $\mathbf{v}$, where the operator

$$-\Delta + \nabla\Phi \cdot \nabla$$

acts diagonally on each component of $\mathbf{v}$.

If in addition to the assumptions (1.9), (1.10) on Φ, g satisfies (1.10) and if we assume that both g and Ψ are compactly supported, one can see that $\mathbf{v}$ is C^∞ and satisfies $\partial^\alpha \mathbf{v}(x) \to 0$ as $|x| \to \infty$ (see Corollary 3 below), and that $\partial_i v^j = \partial_j v^i$ (see Proposition 3.1 in [7]).

Integrating equation

$$\nabla g = (-\Delta + \nabla\Phi \cdot \nabla)\,\mathbf{v} + \text{Hess}\,\Phi\mathbf{v},$$

one sees that $\mathbf{v}$ is a solution of the system

$$g = \langle g\rangle_\Lambda + \mathbf{v} \cdot \nabla\Phi - \text{div}\,\mathbf{v}.$$

If it turns out that 0 is a critical point of Φ, then

$$\langle g\rangle_\Lambda = \text{div}\,\mathbf{v}(0).$$

Thus, the study of the thermodynamic properties of the mean value is reduced to estimating the derivatives of the solution $\mathbf{v}$.

One of the most striking results is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one forms, leading to sophisticated methods for estimating the correlation functions. This formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [5]. The formula may be written as follow:

$$\text{cov}(g, h) = \int \left(A_\Phi^{(1)^{-1}} \nabla g \cdot \nabla h\right) e^{-\Phi(x)} dx. \tag{1.14}$$

Recall that the Brascamp-Lieb inequality states that for an arbitrary function $g \in C^1(\mathbb{R}^\Lambda) \cap L^2(\mu^\Lambda)$, when the given measure $d\mu^\Lambda = e^{-\Phi} dx$ has a real-valued, strictly convex C^2-potential Φ then

$$\text{Var}(g) = \text{cov}(g, g) \le \int \left((\text{Hess}\,\Phi)^{-1} \nabla g \cdot \nabla g\right) e^{-\Phi(x)} dx. \tag{1.15}$$

This is indeed an immediate consequence of formula (1.14). It only suffices to observe that $A_\Phi^{(1)^{-1}} \le (\text{Hess}\,\Phi)^{-1}$ which follows from the fact that $W_\Phi^{(0)} = (-\partial_x + \frac{\nabla\Phi}{2})(\partial_x + \frac{\nabla\Phi}{2})$ is a positive operator.

To understand the idea behind formula (1.14), let us recall that $\langle f\rangle_\Lambda$ denotes the mean value of f with respect to the measure $d\mu^\Lambda$. The covariance of two functions f and g is defined by

$$\operatorname{cov}(g,h)=\big\langle(g-\langle g\rangle_\Lambda)(h-\langle h\rangle_\Lambda)\big\rangle_\Lambda . \tag{1.16}$$

If one wants to have an expression of the covariance in the form

$$\operatorname{cov}(g,h)=\langle\nabla h\cdot\mathbf{w}\rangle_{L^2(\mathbb{R}^n,\mathbb{R}^n;d\mu^\Lambda)}, \tag{1.17}$$

for a suitable vector field $\mathbf{w}$, we get, after observing that $\nabla h=\nabla(h-\langle h\rangle_\Lambda)$, and integrating by parts:

$$\operatorname{cov}(g,h)=\int(h-\langle h\rangle_\Lambda)(\nabla\Phi-\nabla)\cdot\mathbf{w}d\mu^\Lambda. \tag{1.18}$$

This leads to the question of solving the equation

$$g-\langle g\rangle_\Lambda=(\nabla\Phi-\nabla)\cdot\mathbf{w}. \tag{1.19}$$

Now trying to solve this above equation with $\mathbf{w}=\nabla u$, we obtain the equation

$$\begin{cases} g-\langle g\rangle_\Lambda=A_\Phi^{(0)}u,\\ \langle u\rangle_\Lambda=0.\end{cases} \tag{1.20}$$

Assuming for now the existence of a smooth solution, we get by differentiation of this above equation

$$\nabla g=A_\Phi^{(1)}\nabla u \tag{1.21}$$

and the formula is now easy to see.

The proof of the main result is essentially based on the following proposition.

Proposition 2 *Let g be a smooth function with lattice support Γ satisfying*

$$|\partial^\alpha\nabla g|\le C_\alpha\quad\forall\alpha\in\mathbb{Z}_+^{|\Gamma|} \tag{1.22}$$

and Φ satisfy (1.9)–(1.11). If f is the unique C^∞-solution of the equation

$$\begin{cases} -\Delta f+\nabla\Phi\cdot\nabla f=g-\langle g\rangle_{L^2(\mu)},\\ \langle f\rangle_{L^2(\mu)}=0,\end{cases}$$

then

$$\sum_{i\in\Lambda}f_{x_i}^2(x)e^{2\kappa d(i,S_g)}\le C\quad\forall x\in\mathbb{R}^\Lambda$$

where C and κ are positive constants. C could possibly depend on the size of the support of g but does not depend on Λ and f.

Once this proposition is established, the statement of the main result will follow from Cauchy-Schwartz inequality. Indeed, using formula (1.14) for the representation of the covariance, we have

$$|\operatorname{cov}(g,h)|=\left|\left\langle A_\Phi^{(1)^{-1}}\nabla g\cdot\nabla h\right\rangle_\Lambda\right|$$

$$
\begin{aligned}
&= \left|\langle\nabla f\cdot\nabla h\rangle_\Lambda\right| \\
&\le \int \sum_{i\in\Lambda} \left|f_{x_i}(x)e^{\kappa d(i,S_g)}e^{-\kappa d(i,S_g)}h_{x_i}\right| d\mu^\Lambda(x) \\
&\le \int \left(\sum_{i\in\Lambda} f_{x_i}^2(x)e^{2\kappa d(i,S_g)}\right)^{1/2}\left(\sum_{i\in S_h} h_{x_i}^2(x)e^{-2\kappa d(i,S_g)}\right)^{1/2} d\mu^\Lambda(x) \\
&\le \left[\int \sum_{i\in\Lambda} f_{x_i}^2(x)e^{2\kappa d(i,S_g)}d\mu^\Lambda(x)\right]^{1/2}\left[\int \sum_{i\in S_h} h_{x_i}^2(x)e^{-2\kappa d(i,S_g)}d\mu^\Lambda(x)\right]^{1/2} \\
&\le \sqrt{C}\left[\int \sum_{i\in S_h} h_{x_i}^2(x)d\mu^\Lambda(x)\right]^{1/2} e^{-\kappa d(S_h,S_g)}.
\end{aligned}
$$

C is the same as in the statement of Proposition 2.

This paper is organized in seven sections.

In Sect. 2, we give an outline of the operators and equations involved in the Witten Laplacian method.

In Sect. 3, we discuss preliminary results on Hilbert space methods for elliptic PDE's.

In Sect. 4, we shall provide a rigorous discussion based on Hilbert space methods for the solvability of the corresponding Witten Laplacian equations.

In Sect. 5, we illustrate the family of Hamiltonians discussed in Sects. 3 and 4 through an example of the type introduced by Marc Kac [9]. We shall in fact prove that the solution $\mathbf{v} = \nabla u$ of (1.21) satisfies

$$
\lim_{|x|\to\infty} \partial^\alpha \mathbf{v}(x) = 0 \quad \forall\alpha\in\mathbb{Z}_+^{|\Lambda|}
$$

if Φ is of the form $\frac{x^2}{2} + \Psi(x)$ with both Ψ and g compactly supported. This is due to Helffer-Sjöstrand [7]. We shall then use this result in Sect. 6 to establish weighted estimates leading to the exponential decay of the two-point correlation functions. Propositions 1 and 2 will be proved in this section where the assumptions of compact support on Ψ and g will be removed by means of cut-off arguments.

In Sect. 7, we shall apply our methods to the d-dimensional nearest neighbor Kac model, and discuss possible physical implications of our result.

2 The Basic Equation

We shall first establish the solvability of the equation

$$
\begin{cases} A_\Phi^{(0)} v = g - \langle g\rangle_{L^2(\mu)}, \\ \langle v\rangle_{L^2(\mu)} = 0 \end{cases} \tag{2.1}
$$

by means of Hilbert space methods. The method consists in determining an appropriate function space and an operator which is a natural realization of the problem. In this particular problem, the function spaces to be considered are the Sobolev spaces $B_\Phi^k(\mathbb{R}^\Lambda)$ defined by

$$
B_\Phi^k(\mathbb{R}^\Lambda) = \left\{u\in L^2(\mathbb{R}^\Lambda) : Z_\Phi^\ell \partial^\alpha u \in L^2(\mathbb{R}^\Lambda)\forall \ell + |\alpha| \le k\right\},
$$

where

$$Z_{\Phi} = \frac{|\nabla \Phi|}{2}. \tag{2.2}$$

$\alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{Z}_+^{|\Lambda|}$ is a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_{|\Lambda|}$.

These are subspaces of the well known Sobolev spaces $W^{k,2}(\mathbb{R}^{\Lambda})$, $k \in \mathbb{N}$.

The vital tool in the Hilbert space approach to elliptic boundary value problems is the celebrated Lax-Milgram theorem. The essence of the method is the interpretation of the problem as a variational problem involving a bilinear form defined in a natural way by the problem and acting on the appropriately chosen function spaces.

In general, the Hilbert space method for elliptic differential equations uses the compact embedding theorem for Sobolev spaces. This is a fundamental step in the method in order to be able to apply the Fredholm alternative. See [6] or [13]. Since in the context of our problem we are dealing with unbounded domains, the classical results regarding the compactness of the embedding

$$W^{k,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx) \tag{2.3}$$

for suitable Ω are no longer valid. However, In the case that the L^p spaces are taken with respect to the weighted measure $e^{-\Phi} dx$, with a suitable Φ, we have the following result due to J.-M. Kneib and F. Mignot. See Lemma 5 in [10].

Lemma 1 *If Φ satisfies the condition*

$$\exists \theta \in (0, 1) : \lim_{|x| \to \infty} \left(\theta\, |\nabla \Phi(x)|^2 - \Delta \Phi\right) = \infty$$

then

$$H^1(\mu^{\Lambda}) \hookrightarrow L^2(\mathbb{R}^{\Lambda}, d\mu^{\Lambda})$$

is compact.

Here and in the sequel, $d\mu^{\Lambda}$ will denote the Gibbs measure

$$d\mu = Z_{\Lambda}^{-1} e^{-\Phi} dx,$$

$$Z_{\Lambda} = \int_{\mathbb{R}^{\Lambda}} e^{-\Phi} dx,$$

and $H^k(\mu^{\Lambda})$ denotes the weighted Sobolev space

$$H^k(\mu^{\Lambda}) = \left\{u \in L^2(\mathbb{R}^{\Lambda}, d\mu^{\Lambda}) : \partial^{\alpha} u \in L^2(\mathbb{R}^{\Lambda}, d\mu^{\Lambda}) \forall\, |\alpha| \le k\right\}.$$

Proof We shall prove that every bounded sequence in $H^1(\mu^{\Lambda})$ has a convergent subsequence in $L^2(\mathbb{R}^{\Lambda}, d\mu^{\Lambda})$. Let $\{u_k\} \subset H^1(\mu^{\Lambda}) = H^1(\mathbb{R}^{\Lambda}, d\mu^{\Lambda})$ be such that

$$\|u_k\|_{H^1_{\mu}} \le \sqrt{M} \quad \text{for every } k \text{ and some } M > 0.$$

For any $R > 0$, denote by $B(0, R)$ the open ball centered at 0 with radius R. It is clear that $H^1(\mathbb{R}^{\Lambda}, d\mu^{\Lambda}) \subset H^1(B(0, R), d\mu^{\Lambda})$. Hence $\{u_k\}$ is a bounded sequence in $H^1(B(0, R), d\mu^{\Lambda})$. Moreover

$$\int_{B(0,R)} u_k^2 dx + \int_{B(0,R)} |Du_k|\, dx$$
$$\leq C_{\Phi,R}\left[\int_{B(0,R)} u_k^2 e^{-\Phi} dx + \int_{B(0,R)} |Du_k|\, e^{-\Phi} dx\right].$$

This implies that $\{u_k\}$ is a bounded sequence in $H^1(B(0,R))$. Now using the standard Sobolev compactness embedding theorem for bounded domains with nice boundary (see [6]), we get the compactness of the embedding

$$H^1(B(0,R)) \hookrightarrow L^2(B(0,R)).$$

Therefore, one can find a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that u_{k_j} converges in $L^2(B(0,R))$. We shall prove that $\{u_{k_j}\}$ is Cauchy in $L^2(\mathbb{R}^\Lambda, d\mu^\Lambda)$. Let $\eta > 0$. The assumption of the lemma implies that

$$\zeta := |\nabla\Phi|^2 - (1+\eta)\Delta\Phi \tag{2.4}$$

is positive in a neighborhood of ∞ when $\theta = (1+\eta)^{-1}$

$$\int_{\mathbb{R}^\Lambda} |u_{k_j} - u_{k_l}|^2 e^{-\Phi} dx$$
$$\leq \int_{|x|<R} |u_{k_j} - u_{k_l}|^2 e^{-\Phi} dx + \int_{|x|\geq R} \frac{\zeta\, |u_{k_j} - u_{k_l}|^2}{\inf_{\mathbb{R}^\Lambda \setminus B(0,R)} \zeta} e^{-\Phi} dx$$
$$\leq C_\Phi \int_{|x|<R} |u_{k_j} - u_{k_l}|^2 dx + \int_{|x|\geq R} \frac{\zeta\, |u_{k_j} - u_{k_l}|^2}{\inf_{\mathbb{R}^\Lambda \setminus B(0,R)} \zeta} e^{-\Phi} dx. \tag{2.5}$$

To estimate the last term of the right hand side of this last above inequality, let $\varepsilon > 0$ and choose R large enough so that

$$\inf_{\mathbb{R}^\Lambda \setminus B(0,R)} \zeta \geq \frac{4M(2+\eta+\eta^{-1})}{\varepsilon}.$$

Now introduce the vector fields

$$X_j = \partial_j \tag{2.6}$$

and their formal adjoint in $L^2(\mu)$

$$X_j^* = -\partial_j + \Phi_{x_j}, \tag{2.7}$$

one has when $u \in C_0^\infty(\mathbb{R}^\Lambda)$ for their sum and commutator

$$\left(X_j + X_j^*\right) u = \Phi_{x_j} u \tag{2.8}$$

and

$$[X_j, X_j^*]u = \Phi_{x_j x_j} u. \tag{2.9}$$

It is then straightforward to see that

$$\left(\left[X_j, X_j^*\right] u, u\right)_{\mu^\Lambda} = \left\|X_j^* u\right\|^2_{\mu^\Lambda} - \left\|X_j u\right\|^2_{\mu^\Lambda}, \tag{2.10}$$

$$\left\|\left(X_j + X_j^*\right) u\right\|^2_{\mu^\Lambda} \leq \left(1 + \frac{1}{\eta}\right) \left\|X_j u\right\|^2_{\mu^\Lambda} + (1+\eta) \left\|X_j^* u\right\|^2_{\mu^\Lambda}, \quad \forall \varepsilon > 0 \tag{2.11}$$

so that a linear combination of these formulae gives for any $\eta > 0$

$$\begin{aligned}&\left((|\nabla\Phi|^2 - (1+\eta)\Delta\Phi)u, u\right)_{\mu^\Lambda}\\&\quad\leq (2+\eta+\eta^{-1})\left(\|X_1 u\|^2_{\mu^\Lambda} + \cdots + \|X_m u\|^2_{\mu^\Lambda}\right). \qquad (2.12)\end{aligned}$$

Thus,

$$(\zeta u, u)_{\mu^\Lambda} \leq (2+\eta+\eta^{-1})\,\|u\|^2_{H^1(\mu^\Lambda)}. \qquad (2.13)$$

Because $C_0^\infty(\mathbb{R}^\Lambda)$ is dense in $H^1(\mu^\Lambda)$, this inequality is valid for all $u \in H^1(\mu^\Lambda)$. Now applying (2.13) with u replaced by $u_{k_j} - u_{k_l}$, (2.5) gives

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} \left|u_{k_j} - u_{k_l}\right|^2 e^{-\Phi} dx &\leq C_\Phi \int_{|x|<R} \left|u_{k_j} - u_{k_l}\right|^2 dx + \frac{(2+\eta+\eta^{-1})\|u_{k_j} - u_{k_l}\|^2_{H^1(\mu^\Lambda)}}{4M(2+\eta+\eta^{-1})}\varepsilon\\&\leq C_\Phi \int_{|x|<R} \left|u_{k_j} - u_{k_l}\right|^2 dx + \varepsilon.\end{aligned}$$

The result follows from the convergence of the subsequence $\{u_{k_j}\}$ in $L^2(B(0, R))$. □

3 Preliminary Results on Hilbert Space Methods for Elliptic PDE

A bilinear form with domain H, a complex Hilbert space, is a complex-valued function a defined on $H \times H$ which is such that $a(u, v)$ is linear in u and conjugate linear in v. The inner product $(\cdot, \cdot)_H$ on H is clearly a bilinear form; we shall denote it by $1(\cdot, \cdot)$. The form $a + \lambda 1$ will simply be denoted by $a + \lambda$:

$$(a + \lambda)(u, v) = a(u, v) + \lambda(u, v)_H.$$

The adjoint a^* of a is defined by

$$a^*(u, v) = \overline{a(v, u)}$$

and a is said to be symmetric if $a \equiv a^*$, i.e. for all $u, v \in H$

$$a^*(u, v) = \overline{a(v, u)} = a(u, v).$$

A bilinear form is said to be bounded on $H \times H$ if there exists a constant $M > 0$ such that

$$|a(u, v)| \leq M\,\|u\|_H\,\|v\|_H \quad \text{for all } u, v \in H.$$

A bilinear form a is said to be coercive on H if there exists a positive constant $m > 0$ such that

$$|a(u, u)| \geq m\,\|u\|^2_H \quad \text{for all } u \in H.$$

We shall say that a Banach space W is continuously embedded in a Banach space X if there is a bounded operator $E : W \to X$ which is one-to-one. We call E an embedding operator. We shall say that W is densely embedded in X if $R(E)$, the range of E is dense in X; and we shall write

$$W \hookrightarrow^{E}_{ds} X.$$

If X is a Banach Space, the set of all linear conjugate functionals on X shall be denoted by X^* and is called the conjugate space of X^*.

Suppose X, Y, W, Z are Banach spaces such that

$$W \hookrightarrow^{E}_{ds} X \quad \text{and} \quad Y \hookrightarrow^{F}_{ds} Z^*.$$

Let $a(w,z)$ be a bounded bilinear form on $W \times Z$. We can define two linear operators connected with $a(w,z)$. The first which we shall denote by A, is an operator from X to Y. We say that $x \in D(A)$, the domain of A and $Ax = y$ if $x \in R(E)$, $y \in Y$ and

$$a(E^{-1}x, z) = Fy(z), \quad \text{for all } z \in Z.$$

Since $R(F)$ is dense in Z^*, the operator A is well defined. We call A the operator associated with the bilinear form $a(u,v)$.
The second operator, which we denote by $\hat{A}$, is from W to Z^*. We define it as follows. Fix $w \in W$, $a(w,\cdot) \in Z^*$, it is bounded because the bilinear form a is bounded. We define $\hat{A}w$ to be $a(w,\cdot)$. $\hat{A}$ is clearly well defined and will be called the extended linear operator associated with the bilinear form $a(u,v)$. It can be shown that A and $\hat{A}$ are related in the following way:

$$A = F^{-1}\hat{A}E^{-1}.$$

The fundamental tool to investigate the operator $\hat{A}$ is the Lax, Milgram Theorem.

Theorem 2 (Lax Milgram) *Let a be a bounded coercive form on a Hilbert space H_0 with bounds m and M as above. Then for any $F \in H_0^*$, the adjoint of H_0, there exists a $u \in H_0$ such that*

$$a(u,v) = \langle F, v\rangle \quad \textit{for all } v \in H_0.$$

The map $\hat{A} : u \mapsto F$ defined above is a linear bijection of H_0 onto H_0^* and

$$m \le \|\hat{A}\| \le M, \qquad M^{-1} \le \|\hat{A}^{-1}\| \le m^{-1}.$$

Proof See [6]. □

Corollary 1 *For any choice of $F \in H_0^*$ there is a unique vector $u \in H_0$ satisfying*

$$(u,v)_{H_0} = F(v) \quad \textit{for all } v \in H_0,$$

moreover, the isomorphism $\hat{A}^{-1}$ from H_0^* onto H_0 defined by $\hat{A}^{-1}F = u$ verifies

$$\|\hat{A}^{-1}F\|_{H_0} = \|F\|_{H_0^*}.$$

Next, we apply the Lax-Milgram Theorem to the situation where the Hilbert space H_0 is continuously and densely embedded in another Hilbert space H.

Lemma 2 *If H is a Hilbert space and W is a Banach space continuously and densely embedded in H with embedding operator E, then H can be continuously and densely embedded in W^* with embedding operator F satisfying*

$$(x, Ew)_H = Fx(w), \quad x \in H, \textit{ and } w \in W.$$

Proof For each $x \in H$, the function $x^* : w \mapsto (x, Ew)_H$ is a conjugate linear functional on W and

$$|x^*(x)| \leq \|x\|_H \|E\| \|w\|_W.$$

Hence $x^* \in W^*$. Define the operator F from H to W^* by $Fx = x^*$. Clearly, F is linear and bounded. It is also one-to-one since $R(E)$ is dense in H. Finally, suppose $x^*(w) = 0$ for all $x^* \in R(F)$. Then $(x, Ew)_H = 0$ for all $x \in H$. Thus $Ew = 0$ and consequently $w = 0$. This shows that $R(F)$ is dense. □

Now let the Hilbert space H_0 be continuously and densely embedded into another Hilbert space H with embedding operator E. By the lemma above, H can be continuously and densely embedded in H_0^* with embedding operator F. We obtain the scheme

$$H_0 \hookrightarrow^{E}_{ds} H \hookrightarrow^{F}_{ds} H_0^*$$

which is referred to by saying that (H_0, H, H_0^*) is a Hilbert triplet. Notice also that if the embedding E is compact, then so is the embedding

$$H_0 \hookrightarrow^{FE} H_0^*.$$

Returning to the bilinear form on H_0, we weaken the notion of coerciveness as follows: We say that a bilinear form $a(u, v)$ on H_0 is coercive relative to H, if there exists some $\lambda > 0$ such that $a_\lambda(u, v) = a(u, v) + \lambda(u, v)_H$ is coercive, i.e.

$$a(u, u) + \lambda \|u\|_H^2 \geq \alpha_0 \|u\|_{H_0}^2 \quad \text{for } u \in H_0 \text{ and some } \alpha_0 > 0.$$

If this last inequality above holds, then by the Lax-Milgram Theorem, the extended linear operator $\hat{A}_\lambda$ associated with the bilinear form $a_\lambda(u, v)$ has a bounded inverse $\hat{A}_\lambda^{-1} : H_0^* \to H_0$, moreover $\hat{A}_\lambda u = \hat{A}u + \lambda \hat{B}u$, where $\hat{A}$ is the extended operator associated with the bilinear form $a(u, v)$ and $\hat{B}$ the extended operator associated with the inner product $(u, v)_H$.

Now let $q \in H_0^*$ and consider the equation

$$u \in H_0, \quad \hat{A}u = q \tag{3.1}$$

(3.1) can now be written as

$$u \in H_0, \quad u - \lambda \hat{A}_\lambda^{-1} \hat{B}u = z \tag{3.2}$$

with $z = \hat{A}_\lambda^{-1} q$. We now claim that the compactness of the embedding E implies that of the operator $\hat{A}_\lambda^{-1} \hat{B} : H_0 \to H_0$ is compact. Indeed this follows from the fact that $\hat{B}$ is bounded and $\hat{A}_\lambda^{-1} : H_0^* \to H_0$ is compact. By the Fredholm alternative (see Theorem 4 below), (3.2) is uniquely solvable for any choice of $z \in H_0$ if and only if $u = 0$ is the unique vector of H_0 satisfying $u - \lambda \hat{A}_\lambda^{-1} \hat{B}u = 0$. When this is the case, the linear operator $z \longmapsto u$ defined by (3.2) is bounded from H_0 to H_0. Summarizing, we have the following theorem.

Theorem 3 *Let (H_0, H, H_0^*) be a Hilbert triplet with H_0 compactly embedded in H, let $a(u, v)$ be a bounded bilinear form on H_0 coercive relative to H. Then*

$$u \in H_0, \quad a(u, v) = q(v) \quad \textit{for } v \in H_0$$

admits a unique solution u for any choice of $q \in H_0^*$ *if and only if it admits a unique solution* $u = 0$ *for* $q = 0$ *in which case the solution u satisfies*

$$\|u\|_{H_0} \le C\|q\|_{H_0}$$

with C *dependent only on* $\hat{A}$.

Theorem 4 (Fredholm alternative) *Let* T *be a compact linear operator on a Hilbert space* V *and consider the equations*

$$u \in V, \quad u - Tu = f, \tag{3.3}$$

$$v \in V, \quad v^* - T^* v^* = g \tag{3.4}$$

where T^* *the adjoint operator of* T. *Then the following alternative holds*:

(i) *either there exists a unique solution of* (3.3) *and* (3.4) *for any* f *and* g *in* V, *or*
(ii) *the homogeneous equation*

$$u - Tu = 0$$

has nontrivial solutions. In that case the dimension of the null space of $I - T$ *is finite and equals the dimension of the null space* $\mathcal{N}^*$ *of* $I - T^*$. *Furthermore* (3.3) *and* (3.4) *have* (*non unique*) *solutions if and only if*

$$\langle f, v^* \rangle = 0, \quad \forall v \in \mathcal{N}^*$$

and

$$\langle g, v \rangle = 0, \quad \forall v \in \mathcal{N}$$

$\mathcal{N}$ *being the null space of* $I - T$.

Proof See Yosida [15] (*X*–§5). □

4 Solvability and Regularity of the Basic Equation

Theorem 5 *Let* Λ *be a finite domain in* $\mathbb{Z}^d$. *If* Φ *satisfies*

1. $\lim_{|x|\to\infty} |\nabla\Phi(x)| = \infty$.
2. *For some* M, *any* $\partial^\alpha\Phi$ *with* $|\alpha| = M$ *is bounded on* $\mathbb{R}^\Lambda$.
3. *For* $|\alpha| \ge 1$, $|\partial^\alpha\Phi(x)| \le C_\alpha(1 + |\nabla\Phi(x)|^2)^{1/2}$ *for some* $C_\alpha > 0$.
4. $\operatorname{Hess}\Phi \ge \delta$ *for some* $0 < \delta \le 1$,

then for any C^∞*-function* g *satisfying*

$$|D^\alpha g| \le C_\alpha(1 + Z_\Phi)^{q_\alpha} \tag{4.1}$$

where

$$Z_\Phi = \frac{|\nabla\Phi|}{2},$$

$\alpha \in \mathbb{N}^{|\Lambda|}$ *with some* C_α *and some* $q_\alpha > 0$, *there exists a unique* C^∞*-vector field* v *solution of*

$$\begin{cases} A_\Phi^{(0)} v = g - \langle g \rangle_{L^2(\mu^\Lambda)}, \\ \langle v \rangle_{L^2(\mu^\Lambda)} = 0. \end{cases} \tag{4.2}$$

Proof (Existence) We shall work in the unweighted space $L^2(\mathbb{R}^\Lambda)$ and with the Witten-Laplacians ensuing after the unitary transformation. Under the unitary transformation,

$$A_\Phi^{(0)} v = g - \langle g \rangle_{L^2(\mu^\Lambda)} \quad \text{in } \mathbb{R}^\Lambda$$

is equivalent to

$$\mathbf{W}_\Phi^{(0)} u = q \quad \text{in } \mathbb{R}^\Lambda$$

where

$$u = e^{-\Phi/2} v \quad \text{and} \quad q = e^{-\Phi/2}\big(g - \langle g \rangle_{L^2(\mu^\Lambda)}\big) \in L^2(\mathbb{R}^\Lambda).$$

Recall that

$$B_\Phi^k(\mathbb{R}^\Lambda) = \big\{u \in L^2(\mathbb{R}^\Lambda) : Z_\Phi^l \partial^\alpha u \in L^2(\mathbb{R}^\Lambda) \forall l + |\alpha| \le k\big\}$$

where $\partial^\alpha u$ is taken in the distributional sense in $\mathbb{R}^\Lambda$.

Denote by $B_{0,\Phi}^1(\mathbb{R}^\Lambda)$ the closure of $C_0^\infty(\mathbb{R}^\Lambda)$ in $B_\Phi^1(\mathbb{R}^\Lambda)$, and let $\mathbf{b}$ be the bilinear form on $B_{0,\Phi}^1(\mathbb{R}^\Lambda)$ defined by

$$\mathbf{b} : B_{0,\Phi}^1(\mathbb{R}^\Lambda) \times B_{0,\Phi}^1(\mathbb{R}^\Lambda) \to \mathbb{R}$$

with

$$\mathbf{b}(u, w) = \int_{\mathbb{R}^\Lambda} Du \cdot Dw dx + \int_{\mathbb{R}^\Lambda} \left(\frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2}\right) uw dx.$$

Because we have in mind to apply Theorems 3 and 4 above, we need to check boundedness and coerciveness of $\mathbf{b}$.

Boundedness: After observing that

$$\Delta \Phi \le C(1 + |\nabla \Phi|^2)^{1/2} \le C(1 + |\nabla \Phi|^2),$$

it then follows immediately from Cauchy-Schwartz inequality that

$$|\mathbf{b}(u, w)| \le \alpha_0 \|u\|_{B_\Phi^1(\mathbb{R}^\Lambda)} \|w\|_{B_\Phi^1(\mathbb{R}^\Lambda)}$$

for some constant $\alpha_0 > 0$.

Coerciveness:

$$\int_{\mathbb{R}^\Lambda} |Du|^2 dx = \mathbf{b}(u, u) - \int_{\mathbb{R}^\Lambda} \left(\frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2}\right) |u|^2 dx,$$

$$\begin{aligned} \int_{\mathbb{R}^\Lambda} |Du|^2 dx + \int_{\mathbb{R}^\Lambda} |Z_\Phi u|^2 dx &= \mathbf{b}(u, u) + \int_{\mathbb{R}^\Lambda} \frac{\Delta \Phi}{2} |u|^2 dx \\ &\le \mathbf{b}(u, u) + \varepsilon \int_{\mathbb{R}^\Lambda} \frac{(\Delta \Phi)^2}{4} |u|^2 dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}^\Lambda} |u|^2 dx \end{aligned}$$

$$\leq \mathbf{b}(u,u) + C\varepsilon \int_{\mathbb{R}^\Lambda} |Z_\Phi u|^2\, dx + \left(C\varepsilon + \frac{1}{4\varepsilon}\right) \int_{\mathbb{R}^\Lambda} |u|^2\, dx$$

choosing ε such that $C\varepsilon < 1$ and adding $\int_{\mathbb{R}^\Lambda} |u|^2 dx$ on both side of this above inequality, we immediately get

$$\delta \, \|u\|^2_{B^1_\Phi(\mathbb{R}^\Lambda)} \leq \mathbf{b}(u,u) + \gamma \, \|u\|^2_{L^2(\mathbb{R}^\Lambda)} \tag{4.3}$$

for some positive constants δ and γ.

This shows that the bilinear form $\mathbf{b}(u, v)$ is bounded and coercive relative to $L^2(\mathbb{R}^\Lambda)$.

Observe that $B^1_{0,\Phi}(\mathbb{R}^\Lambda)$ is densely embedded into $L^2(\mathbb{R}^\Lambda)$. Now considering the Hilbert triplet

$$\left(B^1_{0,\Phi}(\mathbb{R}^\Lambda), L^2(\mathbb{R}^\Lambda), B^{-1}_{0,\Phi}(\mathbb{R}^\Lambda)\right), \tag{4.4}$$

where $B^{-1}_{0,\Phi}(\mathbb{R}^\Lambda)$ denote the conjugate space of $B^1_{0,\Phi}(\mathbb{R}^\Lambda)$.

We need to check that the embedding

$$B^1_{0,\Phi}(\mathbb{R}^\Lambda) \hookrightarrow L^2(\mathbb{R}^\Lambda)$$

is compact. This follows from Lemma 1 by simply observing that

$$B^1_{0,\Phi}(\mathbb{R}^\Lambda) \subset U_\Phi^{-1}\left(H^1(\mu)\right)$$

and the fact that U_Φ is a unitary operator.

Let $\mathbf{B}_\gamma$ be the bilinear form in $B^1_{0,\Phi}(\mathbb{R}^\Lambda)$ defined by

$$\mathbf{B}_\gamma(u,w) = \mathbf{b}(u,w) + \gamma \, \langle u, w\rangle_{L^2(\mathbb{R}^\Lambda)}$$

and

$$\hat{A}_\gamma : B^1_{0,\Phi}(\mathbb{R}^\Lambda) \to B^{-1}_{0,\Phi}(\mathbb{R}^\Lambda)$$

be the extended linear operator associated with the bilinear form $\mathbf{B}_\gamma(u, w)$. We have

$$\hat{A}_\gamma u = \hat{A}u + \gamma \hat{B}u, \tag{4.5}$$

where $\hat{A}$ and $\hat{B}$ are the bounded bilinear forms associated with $\mathbf{b}$ and $(\cdot,\cdot)_{L^2}$ respectively.

Note that the equation

$$u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda), \quad \hat{A}u = q$$

is the variational interpretation of the equation

$$\mathbf{W}^{(0)}_\Phi u = q \quad \text{in } \mathbb{R}^\Lambda.$$

By Theorem 2 (Lax-Milgram), the boundedness of $\mathbf{B}_\gamma$ and the coercivity condition

$$\mathbf{B}_\gamma(u,u) \geq \delta \, \|u\|^2_{B^1(\mathbb{R}^\Lambda)} \quad \forall u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda)$$

guarantee that A_γ has a bounded inverse

$$\hat{A}_\gamma^{-1} : B^{-1}_{0,\Phi}(\mathbb{R}^\Lambda) \to B^1_{0,\Phi}(\mathbb{R}^\Lambda).$$

Now using the fact that

$$\hat{A}_\gamma u = \hat{A}u + \gamma \hat{B}u,$$

we can write the equation

$$u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda), \quad \hat{A}u = q$$

as

$$u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda), \quad u - \gamma \hat{A}_\gamma^{-1} \hat{B}u = z \tag{4.6}$$

where

$$z = \hat{A}_\gamma^{-1} q. \tag{4.7}$$

As in the preliminary, because the injection

$$B^1_{0,\Phi}(\mathbb{R}^\Lambda) \hookrightarrow L^2(\mathbb{R}^\Lambda)$$

is compact, the operator $\gamma \hat{A}_\gamma^{-1} \hat{B} : B^1_{0,\Phi}(\mathbb{R}^\Lambda) \to B^1_{0,\Phi}(\mathbb{R}^\Lambda)$ is compact. Moreover, the boundedness of $\gamma \hat{A}_\gamma^{-1} \hat{B}$ implies that

$$\begin{aligned}
\left(\gamma \hat{A}_\gamma^{-1} \hat{B}\right)^* &= \left(\gamma \left(\hat{B}_\gamma^{-1} \hat{A}_\gamma\right)^*\right)^{-1} \\
&= \gamma \left(\hat{A}_\gamma^* \left(\hat{B}^{-1}\right)^*\right)^{-1} \\
&= \gamma \left(\hat{A}_\gamma^* \left(\hat{B}^*\right)^{-1}\right)^{-1} \\
&= \gamma \hat{A}_\gamma^{-1} \hat{B}.
\end{aligned} \tag{4.8}$$

Let us also point out that the self-adjointness of $\hat{A}_\gamma$ and $\hat{B}$ follow from the fact that they are both associated with symmetric bilinear forms.
Now observe that

$$\ker(I - \gamma \hat{A}_\gamma^{-1} \hat{B}) \subset \ker \hat{A}. \tag{4.9}$$

We now claim that

$$\ker \hat{A} = \left\{\delta e^{-\Phi/2}, \delta \in \mathbb{R}\right\}. \tag{4.10}$$

Indeed if $\hat{A}u = 0$, then $\mathbf{b}(u, u) = 0$. Hence

$$\left\| \left(\partial_x + \frac{\nabla \Phi}{2}\right) u \right\|^2_{L^2} = 0$$

which would imply that u is a solution of the equation

$$\left(\partial_x + \frac{\nabla \Phi}{2}\right) u = 0.$$

One can then easily see u must be a constant multiple of $e^{-\Phi/2}$. We have in mind to apply the second part of Theorem 4 (Fredholm alternative). This brings us to check orthogonality of q with $\ker(I-\gamma\hat{A}_\gamma^{-1}\hat{B})$. Let $\delta\in\mathbb{R}$,

$$\begin{aligned}\left\langle\delta e^{-\Phi/2},q\right\rangle_{L^2(\mathbb{R}^\Lambda)}&=\int_{\mathbb{R}^\Lambda}\delta e^{-\Phi/2}e^{-\Phi/2}\big(g-\langle g\rangle_{L^2(\mu^\Lambda)}\big)\\&=\delta\big(\langle g\rangle_{L^2(\mu^\Lambda)}-\langle g\rangle_{L^2(\mu^\Lambda)}\big)=0.\end{aligned}\tag{4.11}$$

Hence using part (ii) of Theorem 4, we conclude that the equation

$$\hat{A}u=q\tag{4.12}$$

is solvable therefore

$$A_\Phi^{(0)}v=g-\langle g\rangle_{L^2(\mu^\Lambda)}\tag{4.13}$$

is solvable in the weak sense. To complete the proof of Theorem 4, we need to prove that the L^2-solution constructed above is a classical solution.

Regularity: Next, we shall prove that the weak solutions constructed above are actually classical solutions. The proof is based on the method of difference quotient.

Theorem 6 (B^k-regularity) *Given $q\in B_\Phi^{k-1}(\mathbb{R}^\Lambda)$ for $k=0,1,2,\dots$, a solution $u\in B_{0,\Phi}^1(\mathbb{R}^\Lambda)$ of*

$$\hat{A}u=q\tag{4.14}$$

is an element of $B_\Phi^{k+1}(\mathbb{R}^\Lambda)$ and we have the estimate

$$\|u\|_{B_\Phi^{k+1}(\mathbb{R}^\Lambda)}\le C\left[\left\|\hat{A}u\right\|_{B_\Phi^{k-1}(\mathbb{R}^\Lambda))}+\|u\|_{B_\Phi^k(\mathbb{R}^\Lambda)}\right]\tag{4.15}$$

for all $u\in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$.

Proof We first establish the result when $k=0$. We have

$$\begin{aligned}\left(\frac{\Delta\Phi}{2}u,u\right)_{L^2}&\le\left\|\frac{\Delta\Phi}{2}u\right\|_{L^2(\mathbb{R}^\Lambda)}\|u\|_{L^2(\mathbb{R}^\Lambda)}\\&\le C\,\|u\|_{B_\Phi^1(\mathbb{R}^\Lambda)}\|u\|_{L^2(\mathbb{R}^\Lambda)}\\&\le\varepsilon C\,\|u\|^2_{B_\Phi^1(\mathbb{R}^\Lambda)}+\frac{C}{4\varepsilon}\|u\|^2_{L^2(\mathbb{R}^\Lambda)}.\end{aligned}\tag{4.16}$$

Thus, for $u\in B_{0,\Phi}^1(\mathbb{R}^\Lambda)$,

$$\begin{aligned}\left\langle\hat{A}u,u\right\rangle&=\|Du\|^2_{L^2(\mathbb{R}^\Lambda)}+\big(Z_\Phi^2u,u\big)_{L^2}-\left(\frac{\Delta\Phi}{2}u,u\right)_{L^2}\\&\ge\|Du\|^2_{L^2(\mathbb{R}^\Lambda)}+\|Z_\Phi u\|^2_{L^2(\mathbb{R}^\Lambda)}-\varepsilon C\,\|u\|^2_{B_\Phi^1(\mathbb{R}^\Lambda)}-\frac{C}{4\varepsilon}\|u\|^2_{L^2(\mathbb{R}^\Lambda)}.\end{aligned}$$

Choosing ε such that $\varepsilon C<1$, we get

$$\left\langle\hat{A}u,u\right\rangle\ge C\,\|u\|^2_{B_\Phi^1(\mathbb{R}^\Lambda)}-C\,\|u\|^2_{L^2(\mathbb{R}^\Lambda)}.$$

Hence

$$\begin{aligned}\|u\|^2_{B^1_\Phi(\mathbb{R}^\Lambda)} &\le C\left\langle \hat{A}u, u\right\rangle + C\,\|u\|^2_{L^2(\mathbb{R}^\Lambda)}\\ &\le C\left\|\hat{A}u\right\|_{B^{-1}_\Phi(\mathbb{R}^\Lambda)}\|u\|_{B^1_\Phi(\mathbb{R}^\Lambda)} + C\,\|u\|^2_{L^2(\mathbb{R}^\Lambda)}\\ &\le \frac{C}{4\varepsilon}\left\|\hat{A}u\right\|^2_{B^{-1}_\Phi(\mathbb{R}^\Lambda)} + C\varepsilon\,\|u\|^2_{B^1_\Phi(\mathbb{R}^\Lambda)} + C\,\|u\|^2_{L^2(\mathbb{R}^\Lambda)}.\end{aligned}$$

Again choosing ε appropriately ($\varepsilon C < 1$) we finally get

$$\|u\|^2_{B^1_\Phi(\mathbb{R}^\Lambda)} \le C\left\|\hat{A}u\right\|^2_{B^{-1}_\Phi(\mathbb{R}^\Lambda)} + C\,\|u\|^2_{B^0_\Phi(\mathbb{R}^\Lambda)}.$$

Now assume that for $u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda)$, $\hat{A}u = q \in B^{k-1}_\Phi(\mathbb{R}^\Lambda)$ implies $u \in B^{k+1}_\Phi(\mathbb{R}^\Lambda)$ and that

$$\|u\|_{B^{k+1}_\Phi(\mathbb{R}^\Lambda)} \le C\left[\left\|\hat{A}u\right\|_{B^{k-1}_\Phi(\mathbb{R}^\Lambda))} + \|u\|_{B^k_\Phi(\mathbb{R}^\Lambda)}\right]. \tag{4.17}$$

Suppose now that $u \in B^1_{0,\Phi}(\mathbb{R}^\Lambda)$, $\hat{A}u \in B^k_\Phi(\mathbb{R}^\Lambda)$. So we know that $u \in B^{k+1}_\Phi(\mathbb{R}^\Lambda)$ and we want to establish that $u \in B^{k+2}_\Phi(\mathbb{R}^\Lambda)$.

Because

$$D_i^h u = \frac{u(x+he_i)-u(x)}{h} \in B^{k+1}_\Phi(\mathbb{R}^\Lambda),$$

replacing u by $D_i^h u$ in inequality (4.17) we get

$$\begin{aligned}\left\|D_i^h u\right\|_{B^{k+1}_\Phi(\mathbb{R}^\Lambda)} &\le C\left[\left\|\hat{A}D_i^h u\right\|_{B^{k-1}_\Phi(\mathbb{R}^\Lambda)} + \left\|D_i^h u\right\|_{B^k_\Phi(\mathbb{R}^\Lambda)}\right]\\ &\le C\left[\left\|D_i^{-h}\hat{A}u\right\|_{B^{k-1}_\Phi(\mathbb{R}^\Lambda)} + \left\|uD_i^h X_\Phi\right\|_{B^{k-1}_\Phi(\mathbb{R}^\Lambda))} + \left\|D_i^h u\right\|_{B^k_\Phi(\mathbb{R}^\Lambda)}\right]\end{aligned}$$

where

$$X_\Phi := \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}.$$

Now letting $h \to 0$ and using assumption 3 on Φ we get

$$\|D_i u\|_{B^{k+1}_\Phi(\mathbb{R}^\Lambda)} \le C\left[\left\|\hat{A}u\right\|_{B^k_\Phi(\mathbb{R}^\Lambda)} + \|u\|_{B^k_\Phi(\mathbb{R}^\Lambda))} + \|u\|_{B^{k+1}_\Phi(\mathbb{R}^\Lambda)}\right]$$

it then follows that

$$D_i u \in B^{k+1}_\Phi(\mathbb{R}^\Lambda).$$

It then only remains to prove that $Z^{k+2}_\Phi u \in L^2(\mathbb{R}^\Lambda)$. To see this first observe that

$$Z^2_\Phi u = \hat{A}u + \Delta u + \frac{\Delta\Phi}{2}u. \tag{4.18}$$

The Laplacian here is taken in the distributional sense. Multiplying by Z_Φ^k on both sides of this last equality, we obtain:

$$Z_\Phi^{k+2}u = Z_\Phi^k \hat{A}u + Z_\Phi^k \Delta u + Z_\Phi^k \frac{\Delta\Phi}{2}u. \tag{4.19}$$

The first term of this equality is in $L^2(\mathbb{R}^\Lambda)$ because $\hat{A}u \in B_\Phi^k(\mathbb{R}^\Lambda)$. That the second terms also belongs to $L^2(\mathbb{R}^\Lambda)$ follows from the fact that $D_i u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$. Finally to see that the last term is an element of $L^2(\mathbb{R}^\Lambda)$, we use assumption 3 on Φ to get that

$$\frac{\Delta\Phi}{2} \le C\left(\frac{1}{4} + Z_\Phi^2\right)^{1/2} \le C\left(\frac{1}{2} + Z_\Phi\right), \tag{4.20}$$

and use the fact that $u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$. □

Proposition 3 (C^∞-regularity) *The weak solution u of $W_\Phi^{(0)}u = q$ is an element of $C^\infty(\mathbb{R}^\Lambda)$.*

The proof of this proposition uses the general Sobolev inequalities theorem given below.

Theorem 7 (General Sobolev inequality) *Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1-boundary. Assume $u \in W^{k,p}(U)$ where*

$$W^{k,p}(U) := \left\{u \in L^1_{loc}(\mathbb{R}^n) : \partial^\alpha u \in L^p(\mathbb{R}^n)\ \forall |\alpha| \le k\right\}.$$

If

$$k > \frac{n}{p}$$

then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

Here $C^{k,\alpha}(\bar{U})$ is the Hölder space consisting of all functions $u \in C^k(\bar{U})$ such that

$$\|u\|_{C^{k,\alpha}(\bar{U})} := \sum_{|\beta|\le k} \sup_{x\in U} \left|\partial^\beta u(x)\right| + \sum_{|\beta|=k} \sup_{\substack{x,y\in U \\ x\ne y}} \left|\frac{\partial^\beta u(x) - \partial^\beta u(y)}{|x-y|^\alpha}\right| < \infty.$$

Proof See [6]. □

Proof of Proposition 3 Because $q \in C^\infty(\mathbb{R}^\Lambda)$, we have $u \in B^k_{loc}(\mathbb{R}^\Lambda)\forall k$, which implies $u \in H^k(V)(= W^{k,2}(V))\forall k$ and $\forall V \Subset \mathbb{R}^\Lambda$. Now choose $k \in \mathbb{N}$ such that $k > |\Lambda|$. Then the theorem above implies that $u \in C^{k,\gamma}(\bar{V})$ for some $0 < \gamma < 1$. Consequently, $u \in C^k(V)$ for an arbitrary big enough k and for any $V \Subset \mathbb{R}^\Lambda$. □

Now that we have enough smoothness, we can make the following remark which completes the proof of Theorem 6.

Remark 2 (End of proof of Theorem 5) A simple integration by parts argument shows that u is in fact a strong solution. It satisfies

$$W_{\Phi}^{(0)} u = q$$

pointwise almost everywhere. Using the unitary transformation and taking gradient on both sides of

$$A_{\Phi}^{(0)} v = g - \langle g \rangle_{L^2(\mu^{\Lambda})},$$

we get

$$A_{\Phi}^{(1)} \nabla v = \nabla g.$$

If $\mathbf{q}$ is a smooth vector field satisfying

$$|\partial^{\alpha} \mathbf{q}| \leq C_{\alpha}(1 + Z_{\Phi})^{q_{\alpha}} \quad \text{for some } q_{\alpha} > 0, \tag{4.21}$$

then one can show as above (this time using uniqueness result of the Fredholm alternative) that the equation

$$A_{\Phi}^{(1)} \mathbf{v} = \mathbf{q}$$

has a unique weak solution $A_{\Phi}^{(1)} \nabla v = \nabla g$ would then imply that two solutions of

$$A_{\Phi}^{(0)} v = g - \langle g \rangle_{L^2(\mu^{\Lambda})} \tag{4.22}$$

must differ by a constant. Thus the problem

$$\begin{cases} A_{\Phi}^{(0)} v = g - \langle g \rangle_{L^2(\mu^{\Lambda})}, \\ \langle v \rangle_{L^2(\mu^{\Lambda})} = 0 \end{cases}$$

has a unique solution. This ends the proof of Theorem 5. □

5 The Kac-like Model

In this section, we propose to illustrate the results above through the study of a more specific family of classical unbounded spin model related to Statistical Mechanics, given by

$$\Phi(x) = \Phi_{\Lambda}(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^{\Lambda}. \tag{5.1}$$

Here we have used the notation $x^2 = x \cdot x$.

The model that was originally suggested by M. Kac [9] corresponds to the case that Ψ is given by

$$\Psi(x) = -2 \sum_{i,j \in \Lambda, i \sim j} \ln \cosh \left[\sqrt{\frac{\beta}{2}} \left(x_i + x_j \right) \right]$$

where β is a small positive constant. This case will be studied in more detail in Sect. 7. In this section, we will discuss the asymptotic behavior of the solution of the equation

$$\begin{cases} -\Delta v + \nabla\Phi \cdot \nabla v = g - \langle g\rangle_{L^2(\mu^\Lambda)}, \\ \langle v\rangle_{L^2(\mu^\Lambda)} = 0 \end{cases} \quad \text{in } \mathbb{R}^\Lambda$$

when both Ψ and g are compactly supported. Let us mention that these assumptions will be relaxed later for the purpose of the main result.

Other aspects of this family of potentials are studied in [7] in the one dimensional case.

Under the assumptions

$$|\partial^\alpha \nabla\Psi| \le C_\alpha, \quad \forall\alpha \in \mathbb{N}^{|\Lambda|}, \tag{5.2}$$

$$\text{Hess}\,\Phi \ge \delta > 0, \quad 0 < \delta < 1, \tag{5.3}$$

one can check that Φ satisfies the assumptions 1–4 as required in Theorem 5.

Let g be a smooth function on $\mathbb{R}^\Gamma$ where Γ is a fixed subset. We shall use the notation

$$x_\Sigma = (x_i)_{i\in\Sigma}$$

if Σ is a proper subset of Λ and shall also assume that $S_g = \Gamma$. Now define the function $\tilde{g}$ on $\mathbb{R}^\Lambda$ by

$$\tilde{g}(x) = g(x_\Gamma), \quad x \in \mathbb{R}^\Lambda.$$

If there is no ambiguity we shall identify $\tilde{g}$ with g.

We propose to prove that if in addition to the assumptions above on Φ, the functions Ψ and g are compactly supported and g satisfies,

$$|\partial^\alpha \nabla g| \le C_\alpha, \quad \forall\alpha \in \mathbb{N}^{|\Lambda|},$$

then the solution v of the equation

$$\begin{cases} -\Delta v + \nabla\Phi \cdot \nabla v = g - \langle g\rangle_{L^2(\mu^\Lambda)}, \\ \langle v\rangle_{L^2(\mu^\Lambda)} = 0 \end{cases} \quad \text{in } \mathbb{R}^\Lambda \tag{5.4}$$

constructed in Sect. 4 satisfies

$$\partial^\alpha \nabla v(x) \to 0 \quad \text{as } |x| \to \infty, \ \forall\alpha \in \mathbb{N}^{|\Lambda|}. \tag{5.5}$$

Recall that under a suitable change of variables, the equation

$$A^{(1)}_\Phi \mathbf{v} = \nabla g \tag{5.6}$$

could be written as

$$\left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}\right) \otimes \mathbf{u} + \text{Hess}\,\Phi\mathbf{u} = \mathbf{q} \tag{5.7}$$

where

$$\mathbf{u} = e^{-\Phi/2}\nabla v \quad \text{and} \quad \mathbf{q} = e^{-\Phi/2}\nabla g. \tag{5.8}$$

Let $B_1 = B_{R_1}(0) \subset \mathbb{R}^\Lambda$ denote a large ball centered at zero with radius R_1 and containing the support of Ψ in $\mathbb{R}^\Lambda$. We also consider a ball $B_2 = B_{R_2}(0) \subset \mathbb{R}^\Gamma$ of radius $R_2 > R_1$ containing the support of g in $\mathbb{R}^\Gamma$. The support of $\tilde{g}$ in $\mathbb{R}^\Lambda$ is then contained in the cylinder

$$B = B_2 \times \mathbb{R}^{\Lambda \setminus \Gamma}.$$

Since B contains the support of Ψ, in $B^c = \mathbb{R}^\Lambda \setminus B$ we have

$$\begin{cases} (-\Delta + \frac{x^2}{4} - \frac{m}{2} + \mathbf{I})\mathbf{u} = 0 & \text{in } B^c, \\ \mathbf{u} = \varphi & \text{on } \partial B \text{ (in the trace sense).} \end{cases} \tag{5.9}$$

Here φ is a C^∞-vector field on ∂B and $m = |\Lambda|$.

Since the operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + \mathbf{I} \tag{5.10}$$

acts diagonally on $\mathbf{u}$, we can work component by component and the situation is reduced to the scalar case

$$\begin{cases} (-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1)u = 0 & \text{in } B^c, \\ u = \varphi & \text{on } \partial B \text{ (in the trace sense).} \end{cases} \tag{5.11}$$

Having reduced the problem to a Dirichlet type for the Schrödinger operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1, \tag{5.12}$$

we shall need some results on the decay of eigenfunctions of the corresponding Schrödinger operator. We need the following lemma:

Lemma 3 *The fundamental solution $\mathcal{E} \in \mathcal{S}'(\mathbb{R}^\Lambda)$ of the operator $-\Delta + k^2 (k > 0)$ exists and is unique. It is spherically symmetric, is an element of $C^\infty(\mathbb{R}^\Lambda \setminus \{0\})$ and has the following asymptotics as $|x| \to \infty$:*

$$\mathcal{E}(x) = C|x|^{(\frac{m-1}{2})} e^{-k|x|}(1 + o(1)). \tag{5.13}$$

In the lemma, $\mathcal{S}'(\mathbb{R}^\Lambda)$ denotes the space of tempered distributions on $\mathbb{R}^\Lambda$.

Proof Consider the equation

$$\left(-\Delta + k^2\right) \mathcal{E}(x) = \delta_o(x). \tag{5.14}$$

Taking Fourier transform, we get

$$\widehat{\left(-\Delta + k^2\right) \mathcal{E}(x)} = \widehat{\delta_o(x)} \tag{5.15}$$

equivalently

$$\left(x^2 + k^2\right) \widehat{\mathcal{E}(x)} = (2\pi)^{-m/2} \tag{5.16}$$

which implies

$$\widehat{\mathcal{E}(x)} = \frac{(2\pi)^{-m/2}}{x^2+k^2}. \tag{5.17}$$

The uniqueness and spherical symmetry follow since

$$\mathcal{E}(x) = (2\pi)^{-m/2}\,\widehat{\widehat{\mathcal{E}(x)}}. \tag{5.18}$$

Furthermore, if $x \neq 0$, the smoothness of $\mathcal{E}(x)$ follows from the regularity theory of the elliptic equation as discussed above in Sect. 4.

$$\left(-\Delta + k^2\right)\mathcal{E}(x) = 0 \quad \text{in } \mathbb{R}^\Lambda \backslash \{0\} \tag{5.19}$$

for $x \neq 0$ set $\mathcal{E}(x) = f(r)$ where $f \in C^\infty(\mathbb{R}^+)$ and $r = |x|$. (5.19) becomes

$$-f''(r) - \frac{m-1}{r} f'(r) + k^2 f(r) = 0. \tag{5.20}$$

Set $f(r) = a(r)g(r)$. Plugging this in (5.20) and setting the coefficient of $g'(r)$ equal zero gives

$$2a' + \frac{m-1}{r} a = 0. \tag{5.21}$$

Take

$$a(r) = r^{-\frac{m-1}{2}}.$$

Then

$$f(r) = r^{-\frac{m-1}{2}} g(r)$$

and (5.20) takes the form

$$g''(r) - k^2\left(1 + O\left(\frac{1}{r^2}\right)\right)g(r) = 0. \tag{5.22}$$

Now using classical results on the asymptotics of the solutions of the Schrödinger operator (see [4]), we discover that

$$g_\pm(r) = Ce^{\pm kr}(1 + o(1)). \tag{5.23}$$

Hence the asymptotics of the solutions of (5.20) are

$$f_\pm(r) = Cr^{-\frac{m-1}{2}} e^{\pm kr}(1 + o(1)). \tag{5.24}$$

Since $\mathcal{E}(x) = f(|x|) \in \mathcal{S}'(\mathbb{R}^\Lambda)$, we conclude that $f = f_-$ and the result follows. □

Theorem 8 *Let Ω be any exterior domain in $\mathbb{R}^\Lambda$ containing a neighborhood of infinity with smooth internal boundary. Let the potential $V(x) \in C^\infty(\Omega)$ and satisfy*

$$\lim_{|x|\to\infty} \inf V(x) \geq E \tag{5.25}$$

and let φ *be a smooth solution of the problem*

$$\begin{cases} (-\Delta + V(x))\,\varphi = \lambda\varphi & in\ \Omega, \\ \rho = \psi & on\ \partial\Omega \end{cases} \tag{5.26}$$

where $\lambda < E$ *and* φ *is a smooth function on* $\partial\Omega$. *Then the following estimate holds*:

$$|\varphi(x)| \leq C_\varepsilon e^{-\sqrt{(a-\lambda-\varepsilon)/2}|x|} \tag{5.27}$$

for any $\varepsilon > 0$.

The proof of this theorem uses the following lemma.

Lemma 4 (A maximum principle) *Let* $k > 0$, Σ *an open subset of* $\mathbb{R}^\Lambda$, *and* $u \in C^2(\Sigma)$ *a function such that*

$$\left(-\Delta + k^2\right) u = f \leq 0 \quad in\ \Sigma. \tag{5.28}$$

Then u cannot have a positive maximum in Σ.

Proof If $x_0 \in \Sigma$ is a maximum point and $u(x_0) > 0$, then

$$\Delta u(x_0) \leq 0, \tag{5.29}$$

this contradicts (5.28). □

Proof of Theorem 8 Let φ be a real solution of the equation

$$H\varphi = \lambda\varphi \quad \text{in } \Omega, \tag{5.30}$$

where

$$H = -\Delta + V(x).$$

We obviously have

$$\Delta\left(\varphi^2\right) = 2\Delta\varphi\cdot\varphi + 2\,|\nabla\varphi|^2 \tag{5.31}$$

$H\varphi = \lambda\varphi$ gives $-\Delta\varphi = (\lambda - V(x))\varphi$ which implies

$$\Delta\left(\varphi^2\right) = 2\,(\lambda - V(x))\,\varphi^2 - 2\,|\nabla\varphi|^2 \tag{5.32}$$

adding $2(b-\lambda)\varphi^2$ on both sides of this equality, we obtain

$$[-\Delta + 2(b-\lambda)]\,\varphi^2 = -2\,(V(x) - b)\,\varphi^2 - 2\,|\nabla\varphi|^2\,. \tag{5.33}$$

Choosing $\lambda < b < E$ the right hand side of (5.33) is non-positive for $|x|$ large enough. Now set

$$u(x) = \varphi^2(x) - M\mathcal{E}(x) \tag{5.34}$$

where $\mathcal{E}(x)$ is the fundamental solution of the operator $-\Delta + k^2$ with

$$k = \sqrt{2(b-\lambda)}. \tag{5.35}$$

Choose R so large that $\mathcal{E}(x) > 0$ and $V(x) > b$ for $|x| > R$. Now choose M so large that $u(x) < 0$ on $\{x \in \overline{\Omega} : |x| = R\}$. We shall prove that

$$u(x) \leq 0 \tag{5.36}$$

on $\{x \in \overline{\Omega} : |x| = R\}$ from which the theorem will follow. Substracting from (5.33) the equation

$$[-\Delta + 2(b - \lambda)]\, M\mathcal{E}(x) = 0, \tag{5.37}$$

we find that (5.28) is satisfied for $u(x)$ with

$$f = -2\,(V(x) - b)\,\varphi^2 - 2\,|\nabla\varphi|^2\,, \quad \text{for } |x| \geq R. \tag{5.38}$$

We then apply the maximum principle in each connected component of the subset

$$\Omega_{R,\rho} = \left\{x \in \overline{\Omega} : R \leq |x| \leq \rho\right\} \tag{5.39}$$

to the function

$$u^\varepsilon(x) = \int u(x - y)\eta_\varepsilon(y)dy \tag{5.40}$$

where $\eta_\varepsilon(x) = \varepsilon^{-m}\eta(\frac{x}{\varepsilon})$ and $\eta(x)$ is the mollifier. Recall that $\eta(x)$ is given by

$$\eta(x) = \begin{cases} e^{-(\frac{1}{1-|x|^2})} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We indeed have

$$\left(-\Delta + k^2\right) u^\varepsilon = f^\varepsilon = \int f(x - y)\eta_\varepsilon(y)dy \leq 0 \tag{5.41}$$

$u \in L^1(\mathbb{R}^\Lambda)$ implies that $u^\varepsilon(x) \to 0$ as $|x| \to \infty$. Set

$$M_\rho(\varepsilon) = \max_{\{x \in \overline{\Omega} : |x| = \rho\}} |u^\varepsilon(x)| \tag{5.42}$$

since $u(x) < 0$ for $x \in \{x \in \overline{\Omega} : |x| = R\}$, using the fact that $u^\varepsilon(x) \rightrightarrows u(x)$ as $\varepsilon \to 0$ on $\{x \in \overline{\Omega} : |x| = R\}$, we conclude that $u^\varepsilon(x) < 0$ on $\{x \in \overline{\Omega} : |x| = R\}$ for small ε. It then follows from Lemma 4 that

$$u^\varepsilon(x) \leq M_\rho(\varepsilon) \quad \text{for } x \in \Omega_{R,\rho}. \tag{5.43}$$

Letting $\rho \to \infty$, we get

$$u^\varepsilon(x) \leq 0 \quad \text{for } x \in \overline{\Omega} \text{ and } |x| \geq R. \tag{5.44}$$

Now since

$$u^\varepsilon(x) \rightrightarrows u(x) \quad \text{as } \varepsilon \to 0 \tag{5.45}$$

in every relatively compact subset of $\{x \in \overline{\Omega} : |x| \geq R\}$, it follows that

$$u(x) \leq 0 \quad \text{for } x \in \left\{x \in \overline{\Omega} : |x| \geq R\right\}. \tag{5.46}$$

□

Corollary 2 *If $V(x) \to \infty$ as $|x| \to \infty$, then for any eigenfunction φ of the boundary value problem in Theorem 8 satisfies, the following estimate*

$$|\varphi(x)| \leq C_a e^{-a|x|} \tag{5.47}$$

where $a > 0$ is arbitrary and $C_a > 0$.

Theorem 9 (Helffer-Sjöstrand [7]) *The L^2-solution u of*

$$\text{(E)} \quad \begin{cases} (-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1)u = 0 & \text{in } B^c, \\ u = \varphi & \text{on } \partial B \text{ (in the trace sense)} \end{cases} \tag{5.48}$$

satisfies

$$u(x) = e^{-\frac{x^2}{4}} |x|^{-1/2} h(x) \tag{5.49}$$

where

$$\partial^\beta h(x) = O(|x|^{-|\beta|}) \quad \forall \beta \in \mathbb{N}^m. \tag{5.50}$$

Using the change of variable $v = e^{\Phi/2} u$ and applying this theorem to each component of u, we obtain

Corollary 3 *The L^2-solution $\mathbf{v}$ of the system*

$$(-\Delta + \nabla\Phi \cdot \nabla)\, \mathbf{v} + \text{Hess}\, \Phi \mathbf{v} = \nabla g \quad \text{in } \mathbb{R}^\Lambda \tag{5.51}$$

satisfies

$$\lim_{|x| \to \infty} \partial^\alpha \mathbf{v}(x) = 0 \quad \forall \alpha \in \mathbb{N}^m. \tag{5.52}$$

Proof of Theorem 9 (Sjöstrand [11]) Denote by

$$K : C^\infty(\partial B) \to C^\infty(B^c) \tag{5.53}$$

the operator that assigns each boundary value the corresponding solution. Since by Theorem 8

$$\lim_{|x| \to \infty} u(x) = 0, \tag{5.54}$$

the maximum principle implies that K is monotone increasing. Indeed, $Kg \geq 0$ whenever $g \geq 0$. This implies that the operator K is increasing and that $Kg \leq \sup g$, if $\sup g \geq 0$, $Kg \geq \inf g$ if $\inf g \leq 0$. Let

$$u_0 = K1\, (\geq 0) \tag{5.55}$$

which is a radial function i.e.

$$u_0(x) = u_0(|x|), \tag{5.56}$$

with

$$\left[-\partial_r^2 - \left(\frac{m-1}{r} \right) \partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 \right] u_0(r) = 0, \quad u_0(R) = 1. \tag{5.57}$$

We perform the Liouville's transformation

$$u_0 = r^{-(m-1)/2} f(r) \tag{5.58}$$

to get rid of the term involving ∂_r. We finally get

$$\left[-\partial_r^2 + \frac{r^2}{4} - \frac{(m-1)(m-3)}{4r^2} + 1 - \frac{m}{2}\right] f(r) = 0, \quad f(R) = R^{(m-1)/2} \tag{5.59}$$

which we write in the form

$$\left[-\partial_r^2 + V(r)\right] f(r) = \frac{m}{2} f(r), \quad f(R) = R^{(m-1)/2}, \tag{5.60}$$

where

$$V(r) = \frac{r^2}{4} - \frac{(m-1)(m-3)}{4r^2} + 1 \to \infty \quad \text{as } r \to \infty. \tag{5.61}$$

Since

$$\int_{r_0}^{\infty} \frac{|V'(r)|^2}{|V(r)|^{5/2}} dr < \infty \quad \text{and} \quad \int_{r_0}^{\infty} \frac{|V''(r)|^2}{|V(r)|^{3/2}} dr < \infty \quad \text{for some large } r_0. \tag{5.62}$$

Classical results on Schrödinger operators (see [1]) allow us to get the asymptotics of $f(r)$ as following:

$$f_{\pm}(r) = Cr^{-1/2} e^{\pm\frac{r^2}{4}} (1 + o(1)). \tag{5.63}$$

Now since $u_0 \to 0$ as $r \to \infty$, we conclude that

$$f(r) = f_{-}(r) = Cr^{-1/2} e^{-\frac{r^2}{4}} (1 + o(1)). \tag{5.64}$$

Hence

$$u_0(r) = Cr^{-\frac{m}{2}} e^{-\frac{r^2}{4}} (1 + o(1)) > 0. \tag{5.65}$$

Next, we write

$$u(x) = j(x) u_0(r). \tag{5.66}$$

Let $g \in C^{\infty}(\partial B)$ be strictly positive everywhere and let

$$u = Kg. \tag{5.67}$$

Denote by $g_{\min} = \inf g$ and $g_{\max} = \sup g$. We obviously have

$$g_{\min} u_0 \le u \le g_{\max} u_0. \tag{5.68}$$

Hence,

$$j(x) = \frac{u(x)}{u_0(x)} \tag{5.69}$$

is bounded. Next, we perform a change in polar coordinates (r, θ) by setting $x = r\theta$. Under this change of coordinates, the operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1 \tag{5.70}$$

becomes

$$-\partial_r^2 - \left(\frac{m-1}{r}\right)\partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2}\Delta_\theta \tag{5.71}$$

where Δ_θ is the Laplace-Beltrami operator on S^{m-1}. Since the operator $-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1$ is rotationally invariant and $\partial_\theta^\alpha u$ takes continuously the value $\partial_\theta^\alpha g$ on ∂B, using the fact that each $\partial_\theta^\alpha u$ arises as infinitesimal rotation, we conclude that for every α, $\partial_\theta^\alpha u$ is a solution of the boundary value problem (E) (under the change of coordinates) with

$$\partial_\theta^\alpha u = \partial_\theta^\alpha g \quad \text{on } \partial B. \tag{5.72}$$

Therefore,

$$\partial_\theta^\alpha u = O(1)e^{-\frac{r^2}{4}}, \quad \forall \alpha \in \mathbb{N}^m, \tag{5.73}$$

which implies

$$\partial_\theta^\alpha j = O(1), \quad \forall \alpha \in \mathbb{N}^m. \tag{5.74}$$

Now we need to control some radial derivative of j. In polar coordinates, we have

$$\left[-\partial_r^2 - \left(\frac{m-1}{r}\right)\partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2}\Delta_\theta\right]u_0(r) = 0. \tag{5.75}$$

Write

$$\left[-\partial_r^2 - \left(\frac{m-1}{r}\right)\partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2}\Delta_\theta\right]j(r,\theta)u_0(r) = 0. \tag{5.76}$$

Using (5.57) and the product rule of differentiation, (5.76) becomes

$$\left[\partial_r^2 + \left[2\frac{\partial_r u_0}{u_0} + \left(\frac{m-1}{r}\right)\right]\partial_r\right]j = -r^{-2}\Delta_\theta j. \tag{5.77}$$

Here

$$\partial_\theta^\alpha\left(r^{-2}\Delta_\theta j\right) = O(r^{-2}), \quad \forall \alpha \in \mathbb{N}^m, \tag{5.78}$$

and

$$\frac{\partial_r u_0}{u_0} = -\frac{r}{2} + O\left(\frac{1}{r}\right). \tag{5.79}$$

Thus, (5.77) can be written as

$$\left[\partial_r^2 + \left[-r + O\left(\frac{1}{r}\right)\right]\partial_r\right]j = O(r^{-2}). \tag{5.80}$$

Let

$$\varphi(r) = r + O\left(\frac{1}{r}\right). \tag{5.81}$$

We have

$$\left[\partial_r - f(r)\right]\partial_r j = O(r^{-2}). \tag{5.82}$$

Let

$$F(r) = \int_1^r f(t)dt \sim r^2. \tag{5.83}$$

Solving (5.82), we get

$$\partial_r j = -\int_r^\infty e^{F(r)-F(s)}\left[O(s^{-2})\right]ds + Ce^{F(r)}. \tag{5.84}$$

Since

$$F(r) - F(s) \sim r^2 - s^2 \leq 2r(r-s) \quad \text{for } s \geq r, \tag{5.85}$$

$\partial_r j$ cannot tend to $\pm\infty$ when $r \to \infty$, we conclude that $C = 0$ and

$$\partial_r j = -\int_r^\infty e^{F(r)-F(s)}\left[O(s^{-2})\right]ds = O(r^{-3}). \tag{5.86}$$

More generally, since $\partial_\theta^\alpha j$ is a solution of (5.82) with right hand side

$$-r^{-2}\partial_\theta^\alpha(\Delta_\theta j) = O(r^{-2}), \tag{5.87}$$

using the same argument as above with j replaced by $\partial_\theta^\alpha j$, we have

$$\partial_r\partial_\theta^\alpha j = O(r^{-3}). \tag{5.88}$$

Now differentiating

$$\left[\partial_r - f(r)\right]\partial_r\partial_\theta^\alpha j = O(r^{-2}) \tag{5.89}$$

with respect to r, we get

$$\left[\partial_r - f(r)\right]\partial_r^2\partial_\theta^\alpha j = O(r^{-3}), \tag{5.90}$$

using again the same argument as before, we get

$$\partial_r^2\partial_\theta^\alpha j = O(r^{-4}) \tag{5.91}$$

continuing this way, we finally get

$$\partial_r^k\partial_\theta^\alpha j = O(r^{-2-k}) \quad k = 1, 2, \ldots. \tag{5.92}$$

Going back to x-coordinates, we get

$$\partial^\alpha j(x) = O(|x|^{-|\alpha|}), \quad \forall\alpha \in \mathbb{N}^m, \alpha \neq 0. \tag{5.93}$$

□

6 Weighted Estimates for the Decay of Correlation

In this section, we propose to get estimates suitable for obtaining the decay of the correlation functions. We shall first analyze the case where Ψ and the source term g are compactly supported

6.1 The Compactly Supported Case

We shall assume that Φ is given by

$$\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^\Lambda, \tag{6.1}$$

where

$$\begin{aligned} |\partial^\alpha \nabla \Psi| &\leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \\ \text{Hess}\, \Phi(x) &\geq \delta \quad \text{for some } 0 < \delta < 1. \end{aligned} \tag{6.2}$$

Again g will denote a smooth function on $\mathbb{R}^\Gamma$ with lattice support $S_g = \Gamma$. We shall identify g with $\tilde{g}$ defined on $\mathbb{R}^\Lambda$ and shall assume that

$$|\partial^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}. \tag{6.3}$$

In addition, we shall momentarily assume that Ψ is compactly supported in $\mathbb{R}^\Lambda$ and g is compactly supported in $\mathbb{R}^\Gamma$ but these assumptions will be relaxed later on.

Assume also that there exists $\delta_0 \in (0, 1)$ such that

$$M^{-1}\, \text{Hess}\, \Phi(x) M \geq \delta_0 \quad \text{(in the sense of (1.11))} \tag{6.4}$$

where M is the diagonal matrix

$$M = \left(\delta_{ij} e^{\kappa d(i, S_g)}\right)_{i,j \in \Lambda}$$

for some $\kappa > 0$. Define

$$|x|_{2,\rho} := \left(\sum_{i \in \Lambda} \rho(i)^2 x_i^2\right)^{1\backslash 2}.$$

Let f be the solution of the equation

$$\begin{cases} -\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle_{L^2(\mu^\Lambda)}, \\ \langle f \rangle_{L^2(\mu^\Lambda)} = 0. \end{cases}$$

Let $t_1 = (t_i)_i \in \mathbb{R}^\Lambda$. We have

$$\begin{aligned} \langle \nabla (\nabla \Phi \cdot \nabla f), t_1 \rangle &= \sum_{i,k \in \Lambda} \left(f_{x_i} \Phi_{x_i x_k} t_k + \Phi_{x_i} f_{x_i x_k} t_k\right) \\ &= \langle \nabla f, \text{Hess}\, \Phi t_1 \rangle + \nabla \Phi \cdot \nabla \langle \nabla f, t_1 \rangle. \end{aligned} \tag{6.5}$$

On the other hand,

$$\langle \nabla (\Delta f), t_1 \rangle = \Delta \langle \nabla f, t_1 \rangle.$$

We therefore have

$$\langle \nabla g, t_1 \rangle = (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla f, t_1 \rangle + \langle \nabla f, \text{Hess}\, \Phi t_1 \rangle . \tag{6.6}$$

Because $\nabla f(x) \to 0$ as $|x| \to \infty$, we consider a point x_0 at which

$$|\nabla f(x)|_{2,\rho} = \left(\sum_{i \in \Lambda} \rho(i)^2 f_{x_i}^2(x) \right)^{1\backslash 2}$$

is maximal. If M is the diagonal matrix

$$M = \left(\delta_{ij} \rho(i) \right)$$

we have

$$\langle \nabla g, M t_1 \rangle = (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla f, M t_1 \rangle + \langle \nabla f, \text{Hess}\, \Phi M t_1 \rangle . \tag{6.7}$$

Now choose

$$t_1 = \left(\rho(i) f_{x_i}(x_0) \right)_{i \in \Lambda} .$$

We need the following lemma.

Lemma 5 *Under the assumptions and notations above, the function*

$$x \longmapsto \langle \nabla f(x), M t_1 \rangle$$

achieves its maximum value at x_0.

Proof Let

$$\zeta(x) = \langle \nabla f(x), M t_1 \rangle \tag{6.8}$$

and

$$\pi(x) = |\nabla f(x)|_{2,\rho}^2 . \tag{6.9}$$

Again by the maximum principle, the function $\zeta(x)$ achieves its maximum at some $\bar{x}_0 \in \mathbb{R}^\Lambda$. It is easy to see that x_0 is a critical point for $\zeta(x)$. Moreover, for any $a \in \mathbb{R}^\Lambda$, we have

$$\begin{aligned} &\langle a, \text{Hess}\, \pi(x_0) a \rangle \\ &= 2 \langle a, \text{Hess}\, \zeta(x_0) a \rangle + 2 \sum_{j,k} \left(\sum_i f_{x_i x_j}(x_0) f_{x_i x_k}(x_0) \rho(i)^2 \right) a_j a_k \\ &= 2 \langle a, \text{Hess}\, \zeta(x_0) a \rangle + 2 \sum_i \rho(i)^2 \left(\sum_j f_{x_i x_j}(x_0) \right)^2 . \end{aligned} \tag{6.10}$$

Because $\langle a, \text{Hess}\, \pi(x_0) a \rangle < 0$, we must have $\langle a, \text{Hess}\, \zeta(x_0) a \rangle < 0$ for any $a \in \mathbb{R}^\Lambda$. Thus, x_0 is a local maximum for $\zeta(x)$. Moreover, on one hand, we have

$$\zeta(\bar{x}_0) \geq \zeta(x_0) = \pi(x_0). \tag{6.11}$$

One the other hand, Cauchy-Schwartz gives

$$\begin{aligned}\zeta(\bar{x}_0) &\le [\pi(\bar{x}_0)]^{1/2}\,[\pi(x_0)]^{1/2}\\ &\le \pi(x_0).\end{aligned} \tag{6.12}$$

These last two above inequalities imply

$$\zeta(\bar{x}_0) = \zeta(x_0) \tag{6.13}$$

and the result follows. □

Now using Lemma 5 above, we have

$$(\nabla\Phi\cdot\nabla - \Delta)\,\langle\nabla f(x_0), Mt_1\rangle \ge 0.$$

This, then implies

$$\begin{aligned}\langle\nabla g(x_0), Mt_1\rangle &\ge \langle\nabla f(x_0), \text{Hess}\,\Phi(x_0)Mt_1\rangle\\ &= \left\langle M\nabla f(x_0), M^{-1}\,\text{Hess}\,\Phi(x_0)Mt_1\right\rangle\\ &= \left\langle t_1, M^{-1}\,\text{Hess}\,\Phi(x_0)Mt_1\right\rangle\\ &\ge \delta_0\,|\nabla f(x_0)|^2_{2,\rho}\,.\end{aligned}$$

Thus

$$\begin{aligned}|\nabla f(x_0)|^2_{2,\rho} &\le \frac{1}{\delta_0}\,\langle M\nabla g(x_0), t_1\rangle\\ &= \frac{1}{\delta_0}\,\|M\nabla g(x_0)\|\,|\nabla f(x_0)|_{2,\rho}\,.\end{aligned}$$

We have almost proved the following proposition.

Proposition 4 *Let g be a smooth function satisfying*

$$|\partial^\alpha\nabla g| \le C_\alpha \quad \forall\alpha\in\mathbb{N}^{|\Gamma|} \tag{6.14}$$

and $\Phi = \frac{x^2}{2} + \Psi(x)$ satisfies (1.9)–(1.11). Assume also that both g and Ψ are compactly supported as above. If f is the unique C^∞-solution of the equation

$$\begin{cases}-\Delta f + \nabla\Phi\cdot\nabla f = g - \langle g\rangle_{L^2(\mu^\Lambda)}\,,\\ \langle f\rangle_{L^2(\mu^\Lambda)} = 0,\end{cases}$$

then

$$\sum_{i\in\Lambda} f^2_{x_i}(x)e^{2\kappa d(i,S_g)} \le C \quad \forall x\in\mathbb{R}^\Lambda$$

C and κ are positive constants. C that could possibly depend on the size of the support of g but does not depend on Λ and f.

Proof If

$$|\nabla f(x_0)|_{2,\rho} = 0 \tag{6.15}$$

there is nothing to prove otherwise we have

$$\begin{aligned}\left(\sum_{i\in\Lambda} f_{x_i}^2(x_0)\rho^2(i)\right)^{1/2} &\le \frac{1}{\delta_0}\left(\sum_{i\in\Lambda} g_{x_i}^2(x_0)\rho^2(i)\right)^{1/2}\\ &= \frac{1}{\delta_0}\left(\sum_{i\in S_g} g_{x_i}^2(x_0)e^{2\kappa d(i,S_g)}\right)^{1/2}\\ &\le \frac{1}{\delta_0}\left(\sum_{i\in S_g} g_{x_i}^2(x_0)\right)^{1/2}\end{aligned}$$

and the result follows. □

Corollary 4 *Let g and h be smooth functions on $\mathbb{R}^{\Gamma}$, and $\mathbb{R}^{\Gamma'}$ where Γ and $\Gamma' \subsetneq \Lambda$ with $\Gamma \cap \Gamma' = \varnothing$ denote respectively the support of g and h and assume that g and h satisfy* (1.10). *Then under the assumptions of Proposition* 4, *we have*

$$|\mathrm{cov}(g,h)| \le Ce^{-\kappa d(S_h,S_g)} \tag{6.16}$$

where C and κ are positive constants that do not depend on Λ, but possibly dependent on the size of the supports of g and h.

6.2 Relaxing the Compact Support Assumptions

We propose now to relax the assumptions of compact support made previously on Ψ and g. As before, let M be the diagonal matrix

$$M = \left(\delta_{ij}\rho(i)\right)$$

where ρ is given by

$$\rho(i) = e^{\kappa d(i,S_g)} \tag{6.17}$$

and

$$M^{-1}\,\mathrm{Hess}\,\Phi(x)M \ge \delta_0 \quad \text{for some } 0 < \delta_0 < 1 \text{ in the sense of (1.11)} \tag{6.18}$$

for every M as above. Next, we propose to generalize the results in Proposition 4 without the assumptions of compact support on Ψ and g by means of a family of cutoff functions. Let us introduce as in [7] a family cutoff functions

$$\chi = \chi_\varepsilon \tag{6.19}$$

($\varepsilon \in [0,1]$) in $C_0^\infty(\mathbb{R})$ with value in [0, 1] such that

$$\begin{cases} \chi = 1 & \text{for } |t| \le \varepsilon^{-1},\\ \left|\chi^{(k)}(t)\right| \le C_k \frac{\varepsilon}{|t|^k} & \text{for } k \in \mathbb{N}.\end{cases}$$

We could take for instance

$$\chi_\varepsilon(t) = f(\varepsilon \ln |t|)$$

for a suitable f. We then introduce

$$\Psi_\varepsilon(x) = \chi_\varepsilon(|x|)\Psi, \quad x \in \mathbb{R}^\Lambda \tag{6.20}$$

and

$$g_\varepsilon(x) = \chi_\varepsilon(|x|)g, \quad x \in \mathbb{R}^\Gamma. \tag{6.21}$$

Recall that

$$-\Delta f + \nabla\Phi \cdot \nabla f = g - \langle g \rangle_{L^2(\mu^\Lambda)} \tag{6.22}$$

which implies

$$(-\Delta + \nabla\Phi \cdot \nabla) \otimes \mathbf{v} + \text{Hess}\,\Phi\mathbf{v} = \nabla g \tag{6.23}$$

where

$$\mathbf{v} = \nabla f.$$

Under the transformations

$$\mathbf{v} = e^{-\Phi/2}\mathbf{u} \quad \text{and} \quad \mathbf{q} = e^{-\Phi/2}\nabla g$$

we have

$$\left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}\right) \otimes \mathbf{Iu} + \text{Hess}\,\Phi\mathbf{u} = \mathbf{q} \quad \text{in } \mathbb{R}^\Lambda. \tag{6.24}$$

We first verify that the assumptions on Ψ and g are satisfied by $\Psi_\varepsilon(x)$ and $g_\varepsilon(x)$. Namely

$$|\partial^\alpha \nabla\Psi| \le C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{6.25}$$

$$|\partial^\alpha \nabla g| \le C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{6.26}$$

and

$$M^{-1}\,\text{Hess}\,\Phi M \ge \delta_0 > 0, \quad 0 < \delta_0 < 1 \tag{6.27}$$

M shall still denote the diagonal matrix

$$M = \left(\delta_{ij}\rho(i)\right)_{i,j\in\Lambda}$$

where ρ is a weight function on $\mathbb{R}^\Lambda$ satisfying

$$e^{-\lambda} \le \frac{\rho(i)}{\rho(j)} \le e^\lambda, \quad \text{if } i \sim j \text{ for some } \lambda > 0. \tag{6.28}$$

Using

$$M^{-1}\,\text{Hess}\,\Psi M \ge \delta_0 - 1,$$

we obtain immediately

$$M^{-1}\,\text{Hess}\,\Psi_\varepsilon(x) M \ge (\delta_0 - 1)\,\chi_\varepsilon(|x|) - C\varepsilon \tag{6.29}$$

for all ε and some constant C. To see this, let us first write

$$\chi_\varepsilon = \chi \text{ and } r = |x|$$

so that

$$\Psi_\varepsilon(x) = \chi(r)\Psi(x),$$

$$\frac{\rho(j)}{\rho(i)}\Psi_{\varepsilon x_i x_j} = \frac{1}{r}\frac{\rho(j)}{\rho(i)}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\chi'(r)\Psi + \frac{\rho(j)}{\rho(i)}\frac{x_i x_j}{r^2}\chi''(r)\Psi + \frac{\rho(j)}{\rho(i)}\frac{x_j}{r}\chi'(r)\Psi_{x_j} + \frac{\rho(j)}{\rho(i)}\chi(r)\Psi_{x_i x_j}.$$

Let $a \in \mathbb{R}^\Lambda$,

$$\begin{aligned}
&\left\langle M^{-1}\,\mathrm{Hess}\,\Psi_\varepsilon(x)Ma, a\right\rangle \\
&= \left(\frac{1}{r}\sum_i a_i^2 - \frac{1}{r^3}\sum_{i,j}\frac{\rho(j)}{\rho(i)}a_i a_j x_i x_j\right)\chi'(r)\Psi \\
&\quad + \frac{1}{r^2}\chi''(r)\Psi\sum_{i,j}\frac{\rho(j)}{\rho(i)}a_i a_j x_i x_j + \frac{1}{r}\chi'(r)\sum_{i,j}\frac{\rho(j)}{\rho(i)}a_i a_j x_j\Psi_{x_j} \\
&\quad + \chi(r)\sum_{i,j}\frac{\rho(j)}{\rho(i)}a_i a_j\Psi_{x_i x_j} \\
&\geq -2\frac{a^2}{r}\left|\chi'(r)\Psi(x)\right| - a^2\left|\chi''(r)\Psi(x)\right| - C\left|\chi'(r)\right|a^2 + (\delta_0 - 1)\,\chi(r)a^2 \\
&\geq \left[(\delta_0 - 1)\,\chi(r) - \varepsilon C\right]a^2.
\end{aligned}$$

We conclude that

$$M^{-1}\,\mathrm{Hess}\,\Psi_\varepsilon(x)M \geq (\delta_0 - 1)\,\chi(r) - \varepsilon C$$

for all $\varepsilon > 0$.

It follows that

$$M^{-1}\,\mathrm{Hess}\,\Phi_\varepsilon(x)M \geq \delta_0 - C\varepsilon. \tag{6.30}$$

Now with δ_0 replaced by $\delta_0' = \delta_0 - C\varepsilon$, we see that

$$M^{-1}\,\mathrm{Hess}\,\Phi_\varepsilon(x)M \geq \delta_0', \quad 0 < \delta_0' < 1 \tag{6.31}$$

for ε small enough. (Notice that ε is possibly Λ-depend.) It remains to check the assumptions on g_ε and Ψ_ε. To see that

$$|\partial^\alpha \nabla g_\varepsilon| \leq C + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}, \tag{6.32}$$

we have

$$g_\varepsilon(x) = \chi_\varepsilon(r)g(x), \quad x \in \mathbb{R}^\Gamma.$$

Again let $|\alpha| \geq 1$, using Leibniz's formula, we have

$$|\partial^\alpha g_\varepsilon| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g$$
$$= |\partial^\alpha g| + |g \partial^\alpha \chi_\varepsilon(r)| + \sum_{\substack{\beta<\alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} \left| \partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g \right|.$$

With the assumption $g(0) = 0$, we write

$$|g(x)| \leq \int_0^1 \sum_{j \in \Lambda} \left| x_j g_{x_j}(sx) \right| ds$$
$$\leq \int_0^1 \left(\sum_{j \in \Lambda} x_j^2 \right)^{1/2} \left(\sum_{j \in \Lambda} g_{x_j}^2(sx) \right)^{1/2} ds$$
$$\leq C_g r \tag{6.33}$$

again using the fact that

$$r \partial^\alpha \chi_\varepsilon(r) = \mathcal{O}_\alpha(\varepsilon),$$

we get

$$|g \partial^\alpha \chi_\varepsilon(r)| = \mathcal{O}_{\alpha,\Lambda}(\varepsilon). \tag{6.34}$$

Observe also that

$$\sum_{\substack{\beta<\alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} \left| \partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g \right| = \mathcal{O}_\alpha(\varepsilon) \tag{6.35}$$

it then immediately follows from the assumption on g that

$$|\partial^\alpha \nabla g_\varepsilon| \leq C_\alpha + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}. \tag{6.36}$$

Similarly, one can prove that

$$|\partial^\alpha \nabla \Psi_\varepsilon| \leq C_\alpha + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}. \tag{6.37}$$

Thus Ψ_ε and g_ε are compactly supported and satisfy all the conditions that were previously required on Ψ and g. If $\mathbf{u}_\varepsilon$ denotes the family of solutions corresponding to the family of data Φ_ε and g_ε, one can see that $\mathbf{u}_\varepsilon$ converges to $\mathbf{u}$ in C^∞. The proof which based on regularity estimates is given in detail in [7]. Consequently, the family of solution $\mathbf{v}_\varepsilon = e^{\Phi_\varepsilon} \mathbf{u}_\varepsilon$ converges to $\mathbf{v}$ in C^∞.

Proposition 5 *If* $g(0) = 0$, *then Proposition* 4 *holds without the assumptions of compact support on* Ψ *and* g.

Proof Using Proposition 4 we have

$$\left(\sum_{i \in \Lambda} f_{\varepsilon x_i}^2(x) e^{2\kappa d(i,S_g)} \right)^{1/2} \leq C \left| S_g \right|^{1/2} + \mathcal{O}_\Lambda(\varepsilon) \quad \forall x \in \mathbb{R}^\Lambda.$$

The result follows by taking the limit as $\varepsilon \to 0$. $\square$

Corollary 5 *If $g = x_i$ and $h = x_j$ we get*

$$\left|\mathrm{Cor}^{\Lambda}(i,j)\right| \le C e^{-\kappa d(i,j)}.$$

7 The *d*-Dimensional Kac Model

An example of a non-quadratic model satisfying the assumptions above is given by

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}\left(x_i + x_j\right)\right]$$

with $\beta > 0$ small enough.

Let us now verify that

$$\Phi_{\Lambda}(x) = \frac{x^2}{2} - 2\sum_{i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}\left(x_i + x_j\right)\right]$$

satisfies the required assumptions,

$$\Psi_{x_i} = -2\sum_{j:,j\sim i} \frac{\sqrt{\frac{\beta}{2}}\sinh[\sqrt{\frac{\beta}{2}}(x_i + x_j)]}{\cosh[\sqrt{\frac{\beta}{2}}(x_i + x_j)]},$$

$$\Psi_{x_i x_k} = \begin{cases} -\beta \sum_{j:,j\sim i} \frac{1}{\cosh^2[\sqrt{\frac{\beta}{2}}(x_i + x_j)]} & \text{if } k = i, \\ -\frac{\beta}{\cosh^2[\sqrt{\frac{\beta}{2}}(x_i + x_k)]} & \text{if } k \sim i, \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that

$$\left|\Psi_{x_i}\right| \le 4d\sqrt{\frac{\beta}{2}},$$

$$\left|\Psi_{x_i x_i}\right| \le 2d\beta,$$

and

$$\left|\Psi_{x_i x_k}\right| \le \beta \quad \text{if } k \sim i.$$

Similarly, using the properties of cosh and sinh and the fact that $\sinh t \le \cosh t$ for all t one can see that all derivatives of order greater than or equal to one are bounded. Now we propose to check that for β small enough, the Kac Hamiltonian satisfies

$$M^{-1}\,\mathrm{Hess}\,\Phi(x) M \ge \delta_0$$

for some $\delta_0 \in (0, 1)$ and M as above.
We need the following lemma.

Lemma 6 (Schur's lemma—the R and C bound) *For each rectangular array*

$$(c_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$$

and each pair of sequence $(x_i)_{1\le i\le m}$ *and* $(y_j)_{1\le j\le n}$ *we have the bound*

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_i y_j\right| \le \sqrt{RC}\left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}\left(\sum_{j=1}^{n}\left|y_j\right|^2\right)^{1/2}$$

where R *and* C *are the row sum and column sum maxima defined by*

$$R=\max_i\sum_{j=1}^{n}\left|c_{ij}\right| \quad \textit{and} \quad C=\max_j\sum_{i=1}^{m}\left|c_{ij}\right|.$$

This bound is known as Schur's lemma, but, ironically, it may be the second most famous result with this name. The Schur's decomposition lemma for $n\times n$ matrices is also known under this name. Nevertheless, this inequality is surely the single most commonly used tool for estimating a quadratic form. Going back to the example, we have for any $a=(a_i)_{i\in\Lambda}\in\mathbb{R}^{\Lambda}$,

$$\begin{aligned}
&\left\langle M^{-1}\,\mathrm{Hess}\,\Phi Ma, a\right\rangle\\
&\quad=\sum_{i,j}\Phi_{x_ix_j}\frac{\rho(i)}{\rho(j)}a_ia_j\\
&\quad=\sum_i\Phi_{x_ix_i}a_i^2+\sum_{i\sim j}\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}a_ia_j\\
&\quad\ge(1-2d\beta)\,a^2+\sum_{i\sim j}\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}a_ia_j.
\end{aligned}$$

Now using the Schur's lemma above, we have

$$\begin{aligned}
\left|\sum_{i\sim j}\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}a_ia_j\right| &\le \sum_{i,j}\left|\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}a_ia_j\right|\\
&\le\sqrt{RC}a^2
\end{aligned}$$

where

$$R=\max_i\sum_j\left|\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}\right|$$

and

$$C=\max_j\sum_i\left|\Psi_{x_ix_j}\frac{\rho(i)}{\rho(j)}\right|.$$

To estimate R, observe that

$$\sum_j \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right| = \left| \Psi_{x_i x_i} \right| + \sum_{j:j\sim i} \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right| .$$

Now using the fact that

$$e^{-\kappa} \le \frac{\rho(i)}{\rho(j)} \le e^{\kappa} \quad \text{if } i \sim j$$

we have

$$\sum_j \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right| \le 2d\beta + 2d\beta e^{\kappa} .$$

Hence

$$R \le 2d\beta \left(1 + e^{\kappa}\right) .$$

Similarly, we have

$$C \le 2d\beta \left(1 + e^{\kappa}\right) .$$

Thus,

$$\begin{aligned} \left\langle M^{-1} \operatorname{Hess} \Phi M a, a \right\rangle &\ge \left[(1 - 2d\beta) - 2d\beta \left(1 + e^{\kappa}\right)\right] a^2 \\ &= \left[1 - 2d\beta \left(2 + e^{\kappa}\right)\right] a^2 . \end{aligned}$$

The result follows for $\beta < \frac{1}{2d(2+e^{\kappa})}$.

Hence, if

$$\beta < \frac{1}{4d}$$

there exists $\kappa > 0$ such that $M^{-1} \operatorname{Hess} \Phi M \ge \delta_0$ with $0 < \delta_0 < 1$.

7.1 Physical Implications of the Result

The Kac model

$$\Phi(x) = \frac{x^2}{2} - 2 \sum_{i\sim j} \ln\cosh \left[\sqrt{\frac{\beta}{2}} \left(x_i + x_j\right) \right]$$

is a mean field model introduced by Marc Kac [9] in an effort to study rigorously certain problems of phase transition and in particular to justify the van der Waals theory of liquid-vapor transition. The exact model is analogous to the two dimensional Ising model and constructed as follows:

Let J be an even positive Lipschitz function satisfying

$$\int_{\mathbb{R}} J(r) dr = 2.$$

Define the family $\{J_\gamma\}_{\gamma>0}$ by

$$\forall r \in \mathbb{R}, \quad J_\gamma(r) = \gamma J(\gamma r).$$

The choice made by Kac in [9] consisted of

$$J(r) = e^{-|r|}.$$

For a fixed $\gamma > 0$, one defines an interaction potential $\mathbb{J}_\gamma$ on $\mathbb{Z}^2 \times \mathbb{Z}^2$ by

$$\mathbb{J}_\gamma(k, l, \tilde{k}, \tilde{l}) = J_\gamma(k - \tilde{k})\mathcal{J}(l, \tilde{l})$$

with

$$\mathcal{J}(l, \tilde{l}) = \delta_{l,\tilde{l}} + \frac{1}{2}\left(\delta_{l,\tilde{l}+1} + \delta_{l,\tilde{l}-1}\right).$$

Here $\delta_{i,j}$ is the Kronecker delta function.

Let Λ be a finite subset of $\mathbb{Z}^2$; the Hamiltonian of the configuration $\sigma_\Lambda = (\sigma_i)_{i\in\Lambda} \in \{-1, 1\}^\Lambda$ with boundary condition $\sigma_{\Lambda^c} = (\sigma_i)_{i\in\Lambda^c}$ is given by

$$H_{\Lambda,\gamma}(\sigma_\Lambda/\sigma_{\Lambda^c}) = -\frac{1}{2}\sum_{i,j\in\Lambda} \mathbb{J}_\gamma(i, j)\sigma_i\sigma_j - \sum_{i\in\Lambda, j\in\Lambda^c} \mathbb{J}_\gamma(i, j)\sigma_i\sigma_j.$$

Kac showed in [9] that when

$$J(r) = e^{-|r|},$$

this model may be studied through the transfer operator

$$K_\gamma^m = e^{-\frac{1}{2}\gamma q(x)} e^{\gamma\Delta_m} e^{-\frac{1}{2}\gamma q(x)},$$

where

$$\gamma q(x) = \frac{1}{2}\tanh\left(\frac{\gamma}{2}\right)\sum_{i=1}^m x_i^2 - \sum_{i=1}^m \log\cosh\left[\sqrt{\frac{\gamma\beta}{2}}\,(x_i + x_{i+1})\right],$$

with the convention $x_{m+1} = x_1$. He proved that when γ approaches 0, the behavior of the system only depends on the Kac potential

$$q(x) = \sum_{i=1}^m \frac{x_i^2}{4} - \sum_{i=1}^m \log\cosh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_{i+1})\right].$$

Thus by reducing the two dimensional problem into a one dimensional problem, M. Kac showed that the critical temperature occurs at $\beta_c = \frac{1}{4}$.

Our method allows the study of higher dimensional cases in the mean field approximation. We in fact proved the exponential decay of the two-point correlation function for the higher dimensional mean field model when $\beta < \beta_c = \frac{1}{4d}$. This justifies the existence of a single phase when $\beta < \beta_c = \frac{1}{4d}$. The expansion of this method to include the study of multiple phases is more difficult and is a subject of current investigation. However, knowing that in the $1 + 1$ dimensional case the critical point of the exact model occurs at $\beta_c = \frac{1}{4}$, we may conjecture that in the $d + 1$ dimensional case, the critical point is at $\beta_c = \frac{1}{4d}$.

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